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*71T-B250. CHARLES F. DUNKL and DONALD E. RAMIREZ, University of Virginia, Charlottesville, Virginia 22901. Central Sidon sets of bounded representation type. Preliminary report.

For G an infinite compact group, let $E \subset \hat{G}$ (the dual of G) be a central Sidon set of bounded representation type. Then E is a uniformly approximable central Sidon set. (As one would expect, this result is modelled after its abelian analogue due to S. Drury [C. R. Acad. Sci. Paris Sér. A 271 (1970), 162-163].) This result together with those of C. Cecchini ["Lacunary Fourier series on noncommutative groups," to appear] yields that the union of two Sidon sets in a compact Lie group is a Sidon set. (Received September 7, 1971.)

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Central Sidon sets of bounded representation type

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In a compact Lie group, the union of two Sidon sets is a Sidon set. The proof uses the method of Drury [2] and a result of Cecchini [1].

Let G be a compact group and $\hat{G}$ its dual (we will use our notation from [3, Chapters 7 and 8]). For $\alpha \in \hat{G}$, let $T_\alpha \in \alpha$. Then T_α is a continuous homomorphism of G into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $1 \le i, j \le n_\alpha$. Let $\chi_\alpha(x) = \mathrm{Tr}(T_\alpha(x)) = \sum_{i=1}^{n_\alpha} T_\alpha(x)_{ii}$ (Tr = trace). We call χ_α the character of α.

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Let $\underline{\phi} = \{(\underline{\phi}_{\underline{\alpha}})_{\underline{\alpha}\in\hat{G}} : \underline{\phi}_{\underline{\alpha}} \in \underline{\mathcal{B}}(C^{n_{\underline{\alpha}}}),\ (\underline{\alpha} \in \hat{G})$, and $\sup_{\underline{\alpha}} ||\underline{\phi}_{\underline{\alpha}}||_\infty < \infty\}$, where $||\cdot||_\infty$ denotes the operator norm. The set of all such $\underline{\phi}$ is denoted by $\underline{\mathcal{L}}^\infty(\hat{G})$. It is a Banach algebra under the norm $||\underline{\phi}||_\infty = \sup \{||\underline{\phi}_{\underline{\alpha}}||_\infty : \underline{\alpha} \in \hat{G}\}$ and coordinatewise operations. For $\underline{\mu} \in M(G)$, the Fourier-Stieltjes transform of $\underline{\mu}$, $\hat{\underline{\mu}}$, is a matrix-valued function defined for $\underline{\alpha} \in \hat{G}$ by

$$\underline{\alpha} \mapsto \hat{\underline{\mu}}_{\underline{\alpha}} = \int_G T_{\underline{\alpha}}(x^{-1})\, d\underline{\mu}(x).$$

Note that $\hat{\underline{\mu}} \in \underline{\mathcal{L}}^\infty(\hat{G})$. For an algebra A we let $\mathfrak{Z}A$ denote the center of A. For $\underline{\mu} \in M(G)$, $\underline{\mu} \in \underline{\mathfrak{Z}}M(G)$ if and only if for all $\underline{\alpha}$ $\hat{\underline{\mu}}_{\underline{\alpha}} = c_{\underline{\alpha}} I_{\underline{\alpha}}$, where $I_{\underline{\alpha}}$ denotes the identity operator on $C^{n_{\underline{\alpha}}}$, and so $||\hat{\underline{\mu}}_{\underline{\alpha}}||_\infty = |c_{\underline{\alpha}}|$.

Let $E \subset \hat{G}$. We say that E is a Sidon set if and only if given any $\underline{\phi} \in \underline{\mathcal{L}}^\infty(\hat{G})$, there exists $\underline{\mu} \in M(G)$ such that $\hat{\underline{\mu}}_{\underline{\alpha}} = \underline{\phi}_{\underline{\alpha}}$ for $\underline{\alpha} \in E$. We say that E is a central Sidon set if and only if given any $\underline{\phi} \in \mathfrak{Z}\underline{\mathcal{L}}^\infty(\hat{G})$, there exists $\underline{\mu} \in \mathfrak{Z}M(G)$ such that $\hat{\underline{\mu}}_{\underline{\alpha}} = \underline{\phi}_{\underline{\alpha}}$ for $\underline{\alpha} \in E$.

Let $E \subset \hat{G}$. We define $\underline{\phi}_E \in \underline{\mathcal{L}}^\infty(\hat{G})$ by $(\underline{\phi}_E)_{\underline{\alpha}} = I_{\underline{\alpha}}$ for $\underline{\alpha} \in E$ and $(\underline{\phi}_E)_{\underline{\alpha}} = 0$ for $\underline{\alpha} \notin E$. If $\underline{\phi}_E \in \underline{\mathcal{M}}(\hat{G})$, the closure of $M(G)^\wedge$ in $\underline{\mathcal{L}}^\infty(\hat{G})$, then E is called a uniformly approximable set.

Let $E \subset \hat{G}$. Suppose $\sup\{n_{\underline{\alpha}} : \underline{\alpha} \in E\} < \infty$, then E is called a set of bounded representation type. We show here that a central Sidon set of bounded representation type is a uniformly approximable set. As one would expect, our proof follows closely the proof of Drury [2] given for the abelian case.

For $\underline{\alpha} \in \hat{G}$, we let $\overline{\underline{\alpha}}$ denote the conjugate of $\underline{\alpha}$ (see [3, p. 78]). For $E \subset \hat{G}$, $\overline{E}$ denotes the set $\{\overline{\underline{\alpha}} : \underline{\alpha} \in E\}$. The trivial representation $x \mapsto 1$ from G to T (T the circle group) is denoted by the symbol 1^*. For notational convenience we will sometimes write $\hat{\underline{\mu}}(\underline{\alpha})$ for $\hat{\underline{\mu}}_{\underline{\alpha}}$ $(\underline{\mu} \in M(G), \underline{\alpha} \in \hat{G})$.

Theorem 1. Let $E \subset \hat{G}$. Suppose $E = \overline{E}$ and E is a central Sidon set of bounded representation type. Then E is a uniformly approximable central Sidon set.

Proof. We may assume that the representation $1^* \notin E$. By the weak-$*$ compactness of the norm bounded sets in M(G), it suffices to show for each finite subset F of E with $F = \overline{F}$ that there exists a function $u \in L^1(G)$ with $\hat{u}_{\underline{\alpha}} = I_{\underline{\alpha}}$ for $\underline{\alpha} \in F$, $||\hat{u}_{\underline{\alpha}}||_\infty \leq 1/2$ for $\underline{\alpha} \notin F \cup \{1^*\}$, and $||u||_1 \leq c$ (c a constant depending only on E).

By the closed graph theorem, there exists a constant $B < \infty$ such that given $\underline{\phi} \in \underline{\mathfrak{Z}}\underline{\mathcal{L}}^{\infty}(\hat{G})$ there exists $\underline{\mu} \in \underline{\mathfrak{Z}}M(G)$

with $||\mu|| \leq B||\phi||_\infty$ and $\hat{\mu}_\alpha = \phi_\alpha$ for $\alpha \in E$. Let $M = \sup\{n_\alpha : \alpha \in E\}$, and $d = 4B^2M^4$.

Let Ω be the finite group $\Pi_{k=1}^{K}\{-1,1\}_k$ written multiplicatively where $F = \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$ ($\alpha_i \neq \alpha_j$ for $i \neq j$). For $\omega \in \Omega$, there exists $\mu_\omega \in \mathfrak{Z}M(G)$, $||\mu_\omega|| \leq B$, and $(\mu_\omega)^\wedge(\alpha_k) = \omega_k I_\alpha$, $1 \leq k \leq K$, where $\omega = (\omega_1, \omega_2, \cdots, \omega_K)$. For each $\omega \in \Omega$ and $\alpha \in \hat{G}$, define the complex function $g_\alpha(\omega)$ by $g_\alpha(\omega) I_\alpha = (\mu_\omega)^\wedge_\alpha$, and let $f_\alpha = g_\alpha * g_\alpha$ (convolution on Ω). Thus for $\xi \in \Omega$,

$$f_\alpha(\xi) = \int_\Omega g_\alpha(\xi\lambda^{-1}) g_\alpha(\lambda)\, dm_\Omega(\lambda),$$

where $\alpha \in \hat{G}$ and m_Ω denotes the normalized Haar measure on Ω, $m_\Omega(\Omega)=1$.

Define $\nu_\omega \in \mathfrak{Z}M(G)$ by

$$\nu_\omega = \frac{1}{2^K} \sum_{\lambda\in\Omega} (\mu_{\omega\lambda^{-1}} * \mu_\lambda) \quad .$$

Then $||\nu_\omega|| \leq B^2$, $(\nu_\omega)^\wedge_\alpha = f_\alpha(\omega) I_\alpha$, and in particular $(\nu_\omega)^\wedge(\alpha_k) = \omega_k I$, $1 \leq k \leq K$, (where I denotes the appropriate identity operator). Also

$$||\hat{f}_\alpha||_1 \leq ||\hat{g}_\alpha||_2^2 = ||g_\alpha||_2^2 \leq ||g_\alpha||_\infty^2 \leq B^2,$$

$\alpha \in \hat{G}$.

Now define u_ω a central continuous function on G by

$$u_{\underline{\omega}}(x) = 2d\ \Pi_{k=1}^{K}\ (1 + \underline{\omega}_k\ R(\underline{\alpha}_k, x)/d),$$

$x \in G$, where for $\underline{\beta} \in F = \{\underline{\alpha}_1, \underline{\alpha}_2, \cdots, \underline{\alpha}_K\}$,

$$R(\underline{\beta}, x) = \begin{cases} n_{\underline{\beta}}(\underline{\chi}_{\underline{\beta}}(x) + \overline{\underline{\chi}}_{\underline{\beta}}(x))/2 & ,\text{if } \underline{\beta} \neq \overline{\underline{\beta}} \\ n_{\underline{\beta}}\ \underline{\chi}_{\underline{\beta}}(x)/2 & ,\text{if } \underline{\beta} = \overline{\underline{\beta}}\ . \end{cases}$$

Further $u_{\underline{\omega}}$ is a positive function since for $x \in G$ and $\underline{\beta} \in F$, $|R(\underline{\beta}, x)| \leq n_{\underline{\beta}}^2 \leq M^2 \leq d$ (since $B \geq 1$).

Let $u \in \underline{\mathfrak{Z}}L^1(G)$ be defined by

$$u = \frac{1}{2^K} \sum_{\underline{\omega} \in \underline{\Omega}} (\underline{\nu}_{\underline{\omega}} * u_{\underline{\omega}}) = \int_{\underline{\Omega}} (\underline{\nu}_{\underline{\omega}} * u_{\underline{\omega}}) dm_{\underline{\Omega}}\ .$$

Now

$$||u||_1 \leq \int_{\underline{\Omega}} ||\underline{\nu}_{\underline{\omega}}||\ ||u_{\underline{\omega}}||_1 dm_{\underline{\Omega}}(\underline{\omega})$$

$$\leq B^2 \int_{\underline{\Omega}} ||u_{\underline{\omega}}||_1 dm_{\underline{\Omega}}(\underline{\omega})$$

$$= B^2 \int_G \int_{\underline{\Omega}} u_{\underline{\omega}}(x)\, dm_{\underline{\Omega}}(\omega)\, dm_G(x) = 2dB^2.$$

Also for $\underline{\alpha} \in F$, we have

$$\begin{aligned}
\hat{u}_\alpha &= \int_G T_\alpha(x^{-1})u(x)\,dm_G(x) \\
&= \int_G T_\alpha(x^{-1})\Big(\frac{1}{2^K}\sum_{\omega\in\Omega}(\nu_\omega * u_\omega)(x)\Big)\,dm_G(x) \\
&= \frac{1}{2^K}\sum_{\omega\in\Omega}(\nu_\omega * u_\omega)\hat{\ }_\alpha \\
&= \frac{1}{2^K}\sum_{\omega\in\Omega}(u_\omega)\hat{\ }_\alpha\,(\nu_\omega)\hat{\ }_\alpha \\
&= \Big(\int_\Omega \omega_k\,\frac{1}{n_\alpha}\int_G \chi_\alpha(x^{-1})u_\omega(x)\,dm_G(x)\,dm_\Omega(\omega)\Big)I_\alpha \quad (\text{where } \alpha=\alpha_k,\ 1\le k\le K) \\
&= \Big(\frac{1}{n_\alpha}\int_G \overline{\chi}_\alpha(x)\,2dn_\alpha\chi_\alpha(x)/2d\ dm_G(x)\Big)I_\alpha \\
&= I_\alpha .
\end{aligned}$$

Let $\alpha \in \hat{G}$, then

$$\begin{aligned}
\|\hat{u}_\alpha\|_\infty &= \Big\|\int_\Omega (\nu_\omega)\hat{\ }_\alpha (u_\omega)\hat{\ }_\alpha\,dm_\Omega(\omega)\Big\|_\infty \\
&= \Big\|\int_\Omega f_\alpha(\omega)(u_\omega)\hat{\ }_\alpha\,dm_\Omega(\omega)\Big\|_\infty \\
&\le \|\hat{f}_\alpha\|_1 \sup\Big\{\Big|\int_\Omega (u_\omega)\hat{\ }_\alpha\,\varepsilon(\omega)\,dm_\Omega(\omega)\Big| : \varepsilon\in\hat{\Omega}\Big\}
\end{aligned}$$

(by the inversion formula)

$$\le B^2 \sup\Big\{\Big\|\int_\Omega (u_\omega)\hat{\ }_\alpha\,\varepsilon(\omega)\,dm_\Omega(\omega)\Big\|_\infty : \varepsilon \in \hat{\Omega}\Big\}.$$

Let $\alpha \notin F \cup \{1^*\}$ and $\varepsilon \in \hat{\Omega}$. It now suffices to show that

$$\Big\|\int_\Omega (u_\omega)\hat{\ }_\alpha\,\varepsilon(\omega)\,dm_\Omega(\omega)\Big\|_\infty \le 1/(2B^2).$$

Theorem 2. Let $E \subset \hat{G}$ be a central Sidon set of bounded representation type. Then E is a uniformly approximable central Sidon set.

Proof. Let $H = T \oplus G$ (T the circle group) and thus $\hat{H} = Z \oplus \hat{G}$ (Z the group of integers), [3, p.27]. Let $E_0 = \{(0,\underline{\alpha}) \in \hat{H} : \underline{\alpha} \in E\}$ and $E_1 = \{(1,\underline{\alpha}) \in \hat{H} : \underline{\alpha} \in E\}$. We identify T with the normal subgroup $T \oplus \{e\}$ (e the identity of G) $\subset H$. We define $M_T(H)$ to be the subspace $\{\underline{\mu} \in M(H) : m_T * \underline{\mu} = \underline{\mu}\}$ where m_T denotes the normalized Haar measure of T. The subspace $M_T(H)$ of $M(H)$ is identified naturally with $M(H/T) \simeq M(G)$ (see [3,p.101]). Also m_T is an idempotent in $M(H)$, $(m_T)\hat{}_{\underline{\beta}} = I_{\underline{\beta}}$ for $\underline{\beta} \in T^{\perp} = 0 \oplus \hat{G}$, the annihilator of T, and $(m_T)\hat{}_{\underline{\beta}} = 0$ for $\underline{\beta} \notin T^{\perp}$. It follows that $E_0 = 0 \oplus E$ is a central Sidon set of bounded representation type (in $\hat{H}$). Thus $E_1 = 1 \oplus E$ is a central Sidon set of bounded representation type (in $\hat{H}$).

By repeating the proof of Theorem 1, one shows that there exists $\underline{\mu} \in M(H)$ with $\hat{\underline{\mu}}_{\underline{\alpha}} = I_{\underline{\alpha}}$ for $\underline{\alpha} \in E_1$, and $\|\hat{\underline{\mu}}_{\underline{\alpha}}\|_\infty \le 1/2$ for $\underline{\alpha} \in E_1 \cup \overline{E}_1 \cup \{1^*\}$. Let $\underline{\nu} \in M(H)$ with $\hat{\underline{\nu}}_{\underline{\alpha}} = I_{\underline{\alpha}}$ for $\underline{\alpha} \in \hat{G}_1 = 1 \oplus \hat{G}$ and $\hat{\underline{\nu}}_{\underline{\alpha}} = 0$ for $\underline{\alpha} \notin \hat{G}_1$. Now $\underline{\mu} * \underline{\nu} \in M(H)$ is such that $(\underline{\mu} * \underline{\nu})\hat{}_{\underline{\alpha}} = I_{\underline{\alpha}}$ for $\underline{\alpha} \in E_1$, $(\underline{\mu} * \underline{\nu})\hat{}_{\underline{\alpha}} = 0$ for $\underline{\alpha} \notin \hat{G}_1$ and, $\|(\underline{\mu} * \underline{\nu})\hat{}\|_\infty \le 1/2$ for $\hat{G}_1 \setminus E_1$. Now let $\underline{\sigma} \in M(H)$ be such that $\hat{\underline{\sigma}}(n \oplus \underline{\alpha}) = \hat{\underline{\mu}}((n+1) \oplus \underline{\alpha})$, $n \in Z$, $\underline{\alpha} \in \hat{G}$. Thus $\underline{\sigma} \in M(H)$, $\hat{\underline{\sigma}}_{\underline{\alpha}} = I_{\underline{\alpha}}$ for $\underline{\alpha} \in E_0$, $\hat{\underline{\sigma}}_{\underline{\alpha}} = 0$ for $\underline{\alpha} \notin \hat{G}_0 = 0 \oplus \hat{G}$, and $\|\hat{\underline{\sigma}}_{\underline{\alpha}}\|_\infty \le 1/2$ for $\underline{\alpha} \in \hat{G}_0 \setminus E_0$.

Finally, since $\hat{\underline{\sigma}}$ is supported on $\hat{G}_0$, the annihilator of $T \subset H$, there exists $\underline{\rho} \in M(G) \cong M(H/T)$ with $||\underline{\rho}|| = ||\underline{\sigma}||$ and $\hat{\underline{\rho}}_{\underline{\alpha}} = \hat{\underline{\sigma}}(0 \oplus \underline{\alpha})$, $\underline{\alpha} \in \hat{G}$ (see [3, pp. 99-101]). □

Remark. In a recent paper [1] C. Cecchini has shown that if G is a compact Lie group and E is a $\underline{\Lambda}(4)$ set (for example, if E is a Sidon set [4, p. 423]), then E is a set of bounded representation type.

Theorem 3. Let G be a compact Lie group, then the union of two Sidon sets is a Sidon set.

Proof. One needs only recall that if $M(G)\hat{}\,|E$ is uniformly dense in $\underline{\mathcal{L}}^{\infty}(\hat{G})|E$, then E is a Sidon set, [4, p. 416]. □

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