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MULTIPLIERS ON MODULES OVER THE
 FOURIER ALGEBRA⁽¹⁾

BY

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ABSTRACT. Let G be an infinite compact group and \hat{G} its dual. For $1 \leq p < \infty$, $\mathcal{L}^p(\hat{G})$ is a module over $\mathcal{L}^1(\hat{G}) \cong A(G)$, the Fourier algebra of G . For $1 \leq p, q < \infty$, let $\mathfrak{M}_{p,q} = \text{Hom}_{A(G)}(\mathcal{L}^p(\hat{G}), \mathcal{L}^q(\hat{G}))$. If G is abelian, then $\mathfrak{M}_{p,p}$ is the space of $L^p(\hat{G})$ -multipliers. For $1 \leq p < 2$ and p' the conjugate index of p ,

$$A(G) \cong \mathfrak{M}_{1,1} \subset \mathfrak{M}_{p,p} = \mathfrak{M}_{p',p'} \subsetneq \mathfrak{M}_{2,2} \cong L^\infty(G).$$

Further, the space $\mathfrak{M}_{p,p}$ is the dual of a space called \mathfrak{A}_p , a subspace of $\mathcal{C}_0(\hat{G})$. Using a method of J. F. Price we observe that

$$\bigcup \{ \mathfrak{M}_{q,q} : 1 \leq q < p \} \subsetneq \mathfrak{M}_{p,p} \subsetneq \bigcap \{ \mathfrak{M}_{q,q} : p < q < 2 \}$$

(where $1 < p < 2$). Finally, $\mathfrak{M}_{q,p} = \{0\}$ for $1 \leq p < q < \infty$.

1. Modules over the Fourier algebra. Throughout this paper G will denote an infinite compact group and \hat{G} its dual (we use the notation from [1]). Throughout, $1 \leq p, q, r \leq \infty$. Given p , the conjugate index will be denoted by p' ($1/p + 1/p' = 1$).

Definition. Let $\phi \in \mathcal{C}_F(\hat{G})$ and so $\phi = \hat{f}$ for f a trigonometric polynomial on G . We define $\check{\phi}$ by the rule $\check{\phi} = (\check{f})^\wedge$ where $\check{f}(x) = f(x^{-1})$, $x \in G$.

Proposition 1. The map $\phi \mapsto \check{\phi}$ from $\mathcal{C}_F(\hat{G})$ to $\mathcal{C}_F(\hat{G})$ extends to an isometry of $\mathcal{L}^p(\hat{G})$ ($1 \leq p < \infty$) and of $\mathcal{C}_0(\hat{G})$.

Proof. For f a trigonometric polynomial on G , we have that $(\check{f})^\wedge = ((\bar{f})^*)^\wedge = ((\bar{f})^\wedge)^* = (f\hat{f})^*$ (see [1, p. 87]). Thus for $\phi \in \mathcal{C}_F(\hat{G})$, $\|\check{\phi}\|_p = \|\phi\|_p$. \square

Definition. Let $\phi, \psi \in \mathcal{C}_F(\hat{G})$, we define $\phi \times \psi \in \mathcal{C}_F(\hat{G})$ by the rule $(\phi \times \psi)^\wedge = \hat{\phi}\hat{\psi}$ ($\hat{\phi}$ denotes the inverse Fourier transform of ϕ [1, p. 97]). We note that $\|\phi \times \psi\|_1 \leq \|\phi\|_1 \|\psi\|_1$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$ (see [1, p. 93]). We define the pairing $\langle \phi, \psi \rangle = \text{Tr}(\phi\check{\psi}) = (\check{\phi} * (\check{\psi})^\wedge)(e) = \int_G \hat{\phi}(x)\hat{\psi}(x)dm_G(x)$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$,

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e the identity in G . Equivalently, $\langle \phi, \psi \rangle = (\phi \times \psi)_\iota$ (where ι denotes the trivial representation $x \mapsto 1: G \rightarrow C$).

The map $\langle \phi, \psi \rangle \mapsto \langle \phi, \psi \rangle$ extends to a pairing between $\mathcal{L}^p(\hat{G})$ and $\mathcal{L}^{p'}(\hat{G})$ ($1 \leq p < \infty$), that is, $|\langle \phi, \psi \rangle| \leq \|\phi\|_p \|\psi\|_{p'}$, and $\|\phi\|_p = \sup\{|\langle \phi, \psi \rangle|: \|\psi\|_{p'} \leq 1\}$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$ (see [1, p. 144]).

Theorem 2. For $1/p + 1/q \geq 1$, the map $\langle \phi, \psi \rangle \mapsto \phi \times \psi: \mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G}) \rightarrow \mathcal{C}_F(\hat{G})$ extends to a map of $\mathcal{L}^p(\hat{G}) \times \mathcal{L}^q(\hat{G}) \rightarrow \mathcal{L}^r(\hat{G})$, $1/r = 1/p + 1/q - 1$ (we replace $\mathcal{L}^\infty(\hat{G})$ by $\mathcal{C}_0(\hat{G})$), such that $\|\phi \times \psi\|_r \leq \|\phi\|_p \|\psi\|_q$, $\phi \in \mathcal{L}^p(\hat{G})$, $\psi \in \mathcal{L}^q(\hat{G})$.

Proof. For $\phi, \psi, \theta \in \mathcal{C}_F(\hat{G})$ we define the form F on $\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G})$ by the rule $F(\phi, \psi, \theta) = \langle \phi \times \psi, \theta \rangle = \int_G \hat{\phi}(x)\hat{\psi}(x)\hat{\theta}(x)dm_G(x) = \langle \psi, \phi \times \theta \rangle$; and thus F is symmetric. Now $|F(\phi, \psi, \theta)| \leq \|\phi \times \psi\|_1 \|\theta\|_\infty \leq \|\phi\|_1 \|\psi\|_1 \|\theta\|_\infty$, $\phi, \psi, \theta \in \mathcal{C}_F(\hat{G})$. Let

$$M(a_1, a_2, a_3) = \sup\{|F(\phi_1, \phi_2, \phi_3)|: \phi_j \in \mathcal{C}_F(\hat{G}), \|\phi_j\|_{1/a_j} \leq 1, 1 \leq j \leq 3\},$$

$a_1, a_2, a_3 \in [0, 1]$. By the Riesz-Thorin convexity theorem for integration algebras [1, p. 143], it follows that $\log M$ is a convex function on $[0, 1] \times [0, 1] \times [0, 1]$. Since $M(1, 0, 1), M(1, 1, 0), M(0, 1, 1) \leq 1$, it follows by interpolating that $M(1/p, 1/q, 1/r') \leq 1$ where $1/r = 1/p + 1/q - 1$. \square

Corollary 3. For $1 \leq p < \infty$, $\mathcal{L}^1(\hat{G}) \times \mathcal{L}^p(\hat{G}) = \mathcal{L}^p(\hat{G})$ and so $\mathcal{L}^p(\hat{G})$ is an $\mathcal{L}^1(\hat{G})$ -module. Also $\mathcal{L}^1(\hat{G}) \times \mathcal{C}_0(\hat{G}) = \mathcal{C}_0(\hat{G})$. For $1 < p < \infty$, $\mathcal{L}^p(\hat{G}) \times \mathcal{L}^{p'}(\hat{G}) \subset \mathcal{C}_0(\hat{G})$. For $1/p + 1/q > 1$, $\mathcal{L}^p(\hat{G}) \times \mathcal{L}^q(\hat{G}) \subset \mathcal{L}^r(\hat{G})$, $1/r = 1/p + 1/q - 1$.

Theorem 4. $\mathcal{L}^2(\hat{G}) \times \mathcal{L}^2(\hat{G}) = L^1(G)^\wedge$.

Proof. Let $\phi, \psi \in \mathcal{L}^2(\hat{G})$ and choose $\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty$ sequences of trigonometric polynomials on G such that $\hat{f}_n \xrightarrow{n} \phi, \hat{g}_n \xrightarrow{n} \psi$ in $\mathcal{L}^2(\hat{G})$. Then $f_n g_n \in L^1(G)$, and we wish to show that $\phi \times \psi = \lim_{n \rightarrow \infty} \hat{f}_n \times \hat{g}_n = \lim_{n \rightarrow \infty} (f_n g_n)^\wedge \in L^1(G)^\wedge$. But this follows since $\{f_n g_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^1(G)$.

Conversely, for $b \in L^1(G)$, write $b = fg, f, g \in L^2(G)$. Choose $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty$ sequences from $\mathcal{C}_F(\hat{G})$ such that $\phi_n \xrightarrow{n} \hat{f}, \psi_n \xrightarrow{n} \hat{g}$ in $\mathcal{L}^2(\hat{G})$. Now $\hat{\phi}_n \hat{\psi}_n \xrightarrow{n} fg$ in $L^1(G)$ and so $\hat{b} = (fg)^\wedge = (\lim_{n \rightarrow \infty} \hat{\phi}_n \hat{\psi}_n)^\wedge = \lim_{n \rightarrow \infty} (\hat{\phi}_n \hat{\psi}_n)^\wedge = \lim_{n \rightarrow \infty} \phi_n \times \psi_n = \lim_{n \rightarrow \infty} \phi_n \times \lim_{n \rightarrow \infty} \psi_n \in \mathcal{L}^2(\hat{G}) \times \mathcal{L}^2(\hat{G})$. \square

2. Multipliers on modules over the Fourier algebra.

Definition. Let $1/p + 1/q \geq 1$, $\phi \in \mathcal{L}^p(\hat{G}), \psi \in \mathcal{L}^q(\hat{G})$. We define $\langle \phi, \psi \rangle = (\phi \times \psi)_\iota$. This is an extension of $\langle \cdot, \cdot \rangle$ from $\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G})$.

Definition. Let $1 \leq p, q \leq \infty$. We define $\mathfrak{M}_{p,q} = \text{Hom}_{\mathcal{L}^1(\hat{G})}(\mathcal{L}^p(\hat{G}), \mathcal{L}^q(\hat{G}))$, except that we replace $\mathcal{L}^\infty(\hat{G})$ by $\mathcal{C}_0(\hat{G})$. Note that $\mathcal{L}^p(\hat{G})$ is an $\mathcal{L}^1(\hat{G})$ -module (Corollary 3). (See Rieffel [7] for a more general setting.)

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sup $\{|\langle T\phi, \psi \rangle|: \phi, \psi \in \mathcal{L}^p(\hat{G})\}$ is a convex function.

Proof.

[1, p. 143].

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Proof. T is isomorphic to T^* with multiplicity $f(Tg)$, $f \in A(G)$; that is, there

Theorem

Proof.

$\mathcal{C}_0(\hat{G})$, and T is adjoint of T^* . For $\phi, \psi \in \mathcal{C}_0(\hat{G})$, $\langle \phi, \psi \rangle = \langle \phi \times (T\psi)_\iota, \psi \rangle$.

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Proposition 5. *Let $T: \mathcal{C}_F(\hat{G}) \rightarrow \mathcal{C}_0(\hat{G})$ be a linear map. Define $\|T\|_{p,q} = \sup\{|\langle T\phi, \psi \rangle|: \|\phi\|_p \leq 1, \|\psi\|_{q'} \leq 1, \phi, \psi \in \mathcal{C}_F(\hat{G})\}$. Then $\log\|T\|_{1/a_1, 1/a_2}$ is a convex function for $(a_1, a_2) \in [0, 1] \times [0, 1]$.*

Proof. Apply the Riesz-Thorin convexity theorem for integration algebras [1, p. 143]. \square

Proposition 6. $\mathfrak{M}_{2,2} \cong L^\infty(G)$.

Proof. By taking the inverse Fourier transform we see that $\mathfrak{M}_{2,2}$ is isomorphic to the space of bounded maps T from $L^2(G)$ to $L^2(G)$ which commute with multiplication by elements of $A(G)$, that is, $T: L^2(G) \rightarrow L^2(G)$, $T(fg) = f(Tg)$, $f \in A(G)$, $g \in L^2(G)$. Thus T is multiplication by an element of $L^\infty(G)$, that is, there exists $b \in L^\infty(G)$ such that $Tg = bg$, $g \in L^2(G)$ (let $b = T1$). \square

Theorem 7. *Let $1 \leq p, q \leq \infty$. Then $\mathfrak{M}_{p,q} = \mathfrak{M}_{q',p'}$.*

Proof. We first suppose $1 < p, q < \infty$. Let $T \in \mathfrak{M}_{p,q}$. Thus $T: \mathcal{C}_F(\hat{G}) \rightarrow \mathcal{C}_0(\hat{G})$, and $\|T\|_{p,q} < \infty$. Now $T(\phi \times \psi) = \phi \times (T\psi)$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$. Define the adjoint of T , S by $S: \mathcal{C}_F(\hat{G}) \rightarrow \mathcal{C}_0(\hat{G})$ and $\langle T\phi, \psi \rangle = \langle \phi, S\psi \rangle$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$. For $\phi, \psi \in \mathcal{C}_F(\hat{G})$, $\langle T\phi, \psi \rangle = ((T\phi) \times \psi)_t = (T(\phi \times \psi))_t = (T(\psi \times \phi))_t = ((T\psi) \times \phi)_t = (\phi \times (T\psi))_t = \langle \phi, T\psi \rangle$. Thus S and T agree on $\mathcal{C}_F(\hat{G})$.

Now for $\phi, \psi \in \mathcal{C}_F(\hat{G})$, $\langle T\phi, \psi \rangle = \langle \phi, S\psi \rangle = \langle \phi, T\psi \rangle$, and so $\|T\|_{p,q} = \|T\|_{q',p'}$. It follows that $T|_{\mathcal{C}_F(\hat{G})}$ extends uniquely to an element of $\mathfrak{M}_{q',p'}$ and so $\mathfrak{M}_{p,q} \subset \mathfrak{M}_{q',p'}$. By symmetry $\mathfrak{M}_{q',p'} = \mathfrak{M}_{p,q}$.

We consider now the exceptional cases. Since $\mathcal{L}^1(\hat{G})$ has an identity, we obtain $\mathfrak{M}_{1,p} = \mathcal{L}^p(\hat{G})$ for $1 \leq p < \infty$ and $\mathfrak{M}_{1,\infty} = \mathcal{C}_0(\hat{G})$. Further, applying the previous argument we see that $T \in \mathfrak{M}_{p,\infty}$ implies $T \in \mathfrak{M}_{1,p'} = \mathcal{L}^p(\hat{G})$. But by Corollary 3, $\mathcal{L}^p(\hat{G}) \subset \mathfrak{M}_{p,\infty}$, so $\mathfrak{M}_{p,\infty} = \mathfrak{M}_{1,p'}$. The other spaces $\mathfrak{M}_{p,1}$ ($p > 1$) and $\mathfrak{M}_{\infty,q}$ ($q < \infty$) will be shown to be trivial in Theorem 10. \square

Theorem 8. *Let $1 < p < q < 2$. Then*

$$A(G) \cong \mathcal{L}^1(\hat{G}) = \mathfrak{M}_{1,1} \subset \mathfrak{M}_{p,p} \subset \mathfrak{M}_{q,q} \subset \mathfrak{M}_{2,2} \cong L^\infty(G).$$

Proof. That $\mathcal{L}^1(\hat{G}) = \mathfrak{M}_{1,1}$ follows since $A(G)$ has an identity.

Since $\mathcal{L}^p(\hat{G})$ is an $\mathcal{L}^1(\hat{G})$ -module, $\mathfrak{M}_{1,1} \subset \mathfrak{M}_{p,p}$ (recall Theorem 2).

Let $T \in \mathfrak{M}_{q,q}$. Then $\|T\|_{q,q} = \|T\|_{q',q'} < \infty$. Since $\log\|T\|_{1/a_1, 1/a_2}$ is a convex function of $(a_1, a_2) \in [0, 1] \times [0, 1]$, $\|T\|_{2,2} \leq \|T\|_{q,q}$. Thus $\mathfrak{M}_{q,q} \subset \mathfrak{M}_{2,2}$.

Now for $T \in \mathfrak{M}_{p,p}$, $\|T\|_{p,p} < \infty$. Also $\|T\|_{2,2} \leq \|T\|_{p,p} < \infty$. Now since $1/2 < 1/q < 1/p$, we can interpolate to get $\|T\|_{q,q} \leq \|T\|_{p,p} < \infty$. Thus $\mathfrak{M}_{p,p} \subset \mathfrak{M}_{q,q}$. \square

Theorem 9. *Let $1 \leq p < 2$. Then $\mathfrak{M}_{p,p} \neq \mathfrak{M}_{2,2}$.*

Proof. By way of contradiction, suppose $\mathfrak{M}_{p,p} = \mathfrak{M}_{2,2} = L^\infty(G)^\wedge$. Then $L^\infty(G)^\wedge \subset \mathcal{L}^p(\hat{G})$ (since $\hat{1} \in \mathcal{L}^p(\hat{G})$), and so $\|\hat{f}\|_p \leq C\|f\|_\infty$, $f \in L^\infty(G)$, $C < \infty$. In particular, $f \mapsto \hat{f}$ maps $C_c(G)$ into $\mathcal{L}^p(\hat{G})$, and its adjoint Υ maps $\mathcal{L}^{p'}(\hat{G})$ into $M(G)$. Further $\Upsilon: \mathcal{L}^{p'}(\hat{G}) \rightarrow L^1(G)$ (since $\Upsilon(C_F(\hat{G})) \subset L^1(G)$ and $L^1(G)$ is closed). Let $\phi \in \mathcal{L}^{p'}(\hat{G})$ and $\omega \in \mathcal{L}^\infty(\hat{G})$. Then $\phi\omega \in \mathcal{L}^{p'}(\hat{G})$ and $(\phi\omega)^\wedge \in L^1(G)$, that is, the map $\omega \mapsto (\phi\omega)^\wedge$ takes $\mathcal{L}^\infty(\hat{G})$ into $L^1(G)$. It follows now from a theorem of S. Helgason [5, p. 785] that $\phi \in \mathcal{L}^2(\hat{G})$. Thus $\mathcal{L}^{p'}(\hat{G}) \subset \mathcal{L}^2(\hat{G})$, a contradiction. \square

Theorem 10. *Let $1 \leq p < q \leq \infty$, then $\mathfrak{M}_{q,p} = \{0\}$.*

Proof. First, let $1 < p' < 2 < p$. We show that $\mathfrak{M}_{p,p'} = \{0\}$. For if $T \in \mathfrak{M}_{p,p'}$, $T \neq 0$, then there exists $b \in L^\infty(G)$, $b \neq 0$, such that $f \mapsto bf$ is a bounded linear operator from $L^{p'}(G) \rightarrow L^p(G)$ (consider the maps: $L^{p'}(G) \xrightarrow{\hat{\cdot}} \mathcal{L}^{p'}(\hat{G}) \xrightarrow{\Upsilon} L^{p'}(G) \xrightarrow{b} L^p(G)$, see [1, p. 144]). Thus there exists $C < \infty$ such that $\|bf\|_p \leq C\|f\|_{p'}$, $f \in L^{p'}(G)$. Let $\epsilon > 0$ be such that $\{x: |b(x)| \geq \epsilon\}$ contains a measurable set E with $m_G(E) > 0$, and let χ_E denote the characteristic function of E . Then

$$\epsilon^p m_G(E) \leq \|b\chi_E\|_p^p \leq C^p \|\chi_E\|_{p'}^p = C^p (m_G(E))^{p/p'}$$

and so $0 < \epsilon^p / C^p \leq (m_G(E))^{p/p' - 1}$. But let $m_G(E)$ tend to 0 for the required contradiction. Thus we have established $\mathfrak{M}_{p,p'} = \{0\}$, $1 < p' < 2 < p$.

Now let $T \in \mathfrak{M}_{q,p}$, $T \neq 0$, $1 \leq p < q \leq \infty$, excepting the case $\mathfrak{M}_{\infty,1}$. Thus, $\|T\|_{p',q'} = \|T\|_{q,p} < \infty$. The Riesz-Thorin convexity theorem implies for $1/r = 1/2 - 1/2p + 1/2q$ that $\mathfrak{M}_{r,r'} \neq \{0\}$, a contradiction. Finally, $\mathfrak{M}_{\infty,1} \subset \mathfrak{M}_{2,1} = \{0\}$. \square

Remark. The proof of the above theorem was suggested to us by our colleague John Fournier.

3. Multipliers as dual spaces. For G abelian, $\mathfrak{M}_{p,p}$ is the space of $L^p(\hat{G})$ -multipliers, and A. Figà-Talamanca [4] (also M. Rieffel [7]) has shown it to be a dual space. We now will exhibit this result for the case of G nonabelian (compact). For $p = 1$, $\mathfrak{M}_{1,1}$ is clearly a dual space; indeed, $\mathfrak{M}_{1,1} = \mathcal{L}^1(\hat{G}) = \mathcal{C}_0(\hat{G})^*$ (see [1, p. 88]).

Definition. Let $1 < p \leq 2$. For $\phi \in \mathcal{C}_0(\hat{G})$, we define

$$\|\|\phi\|\|_p = \inf \left\{ \sum_{n=1}^{\infty} \|\phi_n\|_p \|\psi_n\|_{p'} : \phi = \sum_{n=1}^{\infty} \phi_n \times \psi_n \text{ (convergence in } \mathcal{C}_0(\hat{G}) \text{)}, \right. \\ \left. \{\phi_n\}_{n=1}^{\infty} \subset \mathcal{L}^p(\hat{G}), \{\psi_n\}_{n=1}^{\infty} \subset \mathcal{L}^{p'}(\hat{G}) \right\}.$$

We use the convention that $\inf \emptyset = \infty$. The subspace of $\mathcal{C}_0(\hat{G})$ consisting of all ϕ with $\|\|\phi\|\|_p < \infty$ is denoted by \mathfrak{A}_p .

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Remark. By Theorem 4, $\mathfrak{A}_2 = L^1(G)^\wedge$.

Proposition 11. For $1 < p \leq 2$, \mathfrak{A}_p is a Banach space.

Proof. It is easy to show $\|\cdot\|_p$ is a norm. We wish now to show that \mathfrak{A}_p is complete with respect to $\|\cdot\|_p$. Let $\{\phi_n\}_{n=1}^\infty$ be a Cauchy sequence in \mathfrak{A}_p . We may assume that $\|\phi_n - \phi_{n+1}\|_p < 1/2^{n+1}$. Let $\psi_n = \phi_{n+1} - \phi_n \in \mathfrak{A}_p$, and so write ψ_n as $\sum_{m=1}^\infty \theta_{nm} \times \omega_{nm}$, $\theta_{nm} \in \mathfrak{L}^p(\hat{G})$, $\omega_{nm} \in \mathfrak{L}^{p'}(\hat{G})$, and $\sum_{m=1}^\infty \|\theta_{nm}\|_p \|\omega_{nm}\|_{p'} < 1/2^n$. Let $\phi = \phi_1 + \sum_{n=1}^\infty \psi_n$. Now $\|\phi\|_p \leq \|\phi_1\|_p + \sum_{n=1}^\infty 1/2^n < \infty$, and so $\phi \in \mathfrak{A}_p$. Also $\|\phi_m - \phi\|_p = \|\sum_{n=m+1}^\infty \psi_n\|_p < \sum_{n=m+1}^\infty 1/2^n$, which is small for large enough m . \square

Theorem 12. Let $\xi \in \mathfrak{A}_p^*$ ($1 < p \leq 2$). Then there exists $T \in \mathfrak{M}_{p,p}$ such that $\|T\|_{p,p} \leq \|\xi\|$ and $\langle T\phi, \psi \rangle = \xi(\phi \times \psi)$, $\phi, \psi \in \mathcal{C}_F(\hat{G})$.

Proof. For $\phi, \psi \in \mathcal{C}_F(\hat{G})$, $|\xi(\phi \times \psi)| \leq \|\phi \times \psi\|_p \|\xi\| \leq \|\phi\|_p \|\psi\|_{p'} \|\xi\|$. Thus, for each $\phi \in \mathcal{C}_F(\hat{G})$, the map $\psi \mapsto \xi(\phi \times \psi)$ extends to a bounded linear functional on $\mathfrak{L}^{p'}(\hat{G})$. Let $\omega \in \mathfrak{L}^p(\hat{G}) = (\mathfrak{L}^{p'}(\hat{G}))^*$ be such that $\langle \omega, \psi \rangle = \xi(\phi \times \psi)$. Define $T\phi = \omega(\phi \in \mathcal{C}_F(\hat{G}))$. Thus $\langle T\phi, \psi \rangle = \xi(\phi \times \psi)$. Now $T: \mathcal{C}_F(\hat{G}) \rightarrow \mathfrak{L}^p(\hat{G})$ and $\|T\|_{p,p} \leq \|\xi\|$, so we may extend T to all of $\mathfrak{L}^p(\hat{G})$. Finally, to see that $T \in \mathfrak{M}_{p,p}$ we note that $\langle T(\phi_1 \times \phi_2), \psi \rangle = \xi((\phi_1 \times \phi_2) \times \psi) = \xi(\phi_1 \times (\phi_2 \times \psi)) = ((T\phi_1) \times (\phi_2 \times \psi))_i = ((T\phi_1) \times \phi_2, \psi)$, $\phi_1, \phi_2, \psi \in \mathcal{C}_F(\hat{G})$. Thus $T(\phi_1 \times \phi_2) = (T\phi_1) \times \phi_2$, $\phi_1, \phi_2 \in \mathcal{C}_F(\hat{G})$. Thus $T \in \mathfrak{M}_{p,p}$. \square

Proposition 13. Let $\phi \in \mathfrak{L}^p(\hat{G})$ ($1 \leq p < \infty$) or $\phi \in \mathcal{C}_0(\hat{G})$ ($p = \infty$) and $\epsilon > 0$. Then there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\sum_{n=1}^\infty \|\phi_n\|_p < \|\phi\|_p + \epsilon$ and $\sum_{n=1}^\infty \phi_n = \phi$ (convergence in norm).

Proof. For $n = 1, 2, \dots$, let $\psi_n \in \mathcal{C}_F(\hat{G})$ be such that $\|\psi_n - \phi\|_p < \epsilon/2^n$. Let $\phi_1 = \psi_1$ and, for $n = 2, 3, \dots$, let $\phi_n = \psi_{n+1} - \psi_n$. Then $\{\phi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$, $\sum_{n=1}^\infty \|\phi_n\|_p < \|\phi\|_p + \sum_{n=1}^\infty \epsilon/2^n = \|\phi\|_p + \epsilon$, and $\sum_{n=1}^N \phi_n = \psi_{N+1} \xrightarrow{N} \phi$ in $\mathfrak{L}^p(\hat{G})$. \square

Proposition 14. Let $\phi \in \mathfrak{L}^p(\hat{G})$, $\psi \in \mathfrak{L}^{p'}(\hat{G})$, and $\epsilon > 0$ ($1 < p \leq 2$). Then there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\sum_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \|\phi\|_p \|\psi\|_{p'} + \epsilon$, and $\sum_{n=1}^\infty \phi_n \times \psi_n = \phi \times \psi$ (convergence in $\mathcal{C}_0(\hat{G})$).

Proof. Let $\epsilon', \epsilon'' > 0$ be chosen in a way to be specified later. By Proposition 13, there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\sum_{n=1}^\infty \phi_n = \phi$, $\sum_{n=1}^\infty \psi_n = \psi$, $\sum_{n=1}^\infty \|\phi_n\|_p < \|\phi\|_p + \epsilon'$, and $\sum_{n=1}^\infty \|\psi_n\|_{p'} < \|\psi\|_{p'} + \epsilon''$. Let $\phi'_n = \sum_{k=1}^n \phi_k$ and $\psi'_n = \sum_{k=1}^n \psi_k$. Now $\phi'_n \times \psi'_n \xrightarrow{n} \phi \times \psi$ in $\mathcal{C}_0(\hat{G})$ (by joint continuity). Now $\phi'_n \times \psi'_n = \sum_{k,l=1}^n \phi_k \times \psi_l$; also $\sum_{k,l=1}^n \|\phi_k\|_p \|\psi_l\|_{p'} = \sum_{k=1}^n \|\phi_k\|_p \sum_{l=1}^n \|\psi_l\|_{p'} < (\|\phi\|_p + \epsilon')(\|\psi\|_{p'} + \epsilon'') < \|\phi\|_p \|\psi\|_{p'} + \epsilon$ for the

appropriate choice of ϵ', ϵ'' . Finally, note that $\phi'_n \times \psi'_n = \sum_{k,l=1}^n \phi_k \times \psi_l$. \square

Proposition 15. Let $\omega \in \mathcal{A}_p$ ($1 < p \leq 2$) and $\epsilon > 0$. Then there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\omega = \sum_{n=1}^\infty \phi_n \times \psi_n$ (convergence in $\mathcal{C}_0(\hat{G})$) and $\sum_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \|\omega\|_p + \epsilon$.

Proof. There exist sequences $\{\phi'_n\}_{n=1}^\infty \subset \mathcal{L}^p(\hat{G})$ and $\{\psi'_n\}_{n=1}^\infty \subset \mathcal{L}^{p'}(\hat{G})$ such that $\omega = \sum_{n=1}^\infty \phi'_n \times \psi'_n$ and $\sum_{n=1}^\infty \|\phi'_n\|_p \|\psi'_n\|_{p'} < \|\omega\|_p + \epsilon/2$. For each $n = 1, 2, \dots$, there exist sequences $\{\phi_{nm}\}_{m=1}^\infty, \{\psi_{nm}\}_{m=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\phi'_n \times \psi'_n = \sum_{m=1}^\infty \phi_{nm} \times \psi_{nm}$ and $\sum_{m=1}^\infty \|\phi_{nm}\|_p \|\psi_{nm}\|_{p'} < \|\phi'_n\|_p \|\psi'_n\|_{p'} + \epsilon/2^{n+1}$. Now $\sum_{n=1}^\infty \sum_{m=1}^\infty \|\phi_{nm}\|_p \|\psi_{nm}\|_{p'} < \sum_{n=1}^\infty \|\phi'_n\|_p \|\psi'_n\|_{p'} + \epsilon/2 < \|\omega\|_p + \epsilon$ and $\sum_{n=1}^\infty \sum_{m=1}^\infty \phi_{nm} \times \psi_{nm} = \omega$. \square

Proposition 16. Let $\delta > 0$ and let $X_\delta = \{\omega \in \mathcal{C}_F(\hat{G}) : \omega = \sum_{n=1}^N \phi_n \times \psi_n, \phi_n, \psi_n \in \mathcal{C}_F(\hat{G}), \|\omega\|_p + \delta > \sum_{n=1}^N \|\phi_n\|_p \|\psi_n\|_{p'}, \text{ some } N = 1, 2, \dots\}$. Then each X_δ is dense in \mathcal{A}_p ($1 < p \leq 2$).

Proof. Fix $\delta > 0, \xi \in \mathcal{A}_p$, and $0 < \epsilon < \delta/2$. By Proposition 15, there exist sequences $\{\phi_n\}_{n=1}^\infty, \{\psi_n\}_{n=1}^\infty \subset \mathcal{C}_F(\hat{G})$ such that $\xi = \sum_{n=1}^\infty \phi_n \times \psi_n$ and $\sum_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \|\xi\|_p + \epsilon$. Choose N such that $\sum_{n=N+1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \epsilon$ and let $\omega = \sum_{n=1}^N \phi_n \times \psi_n$. Then $\|\omega - \xi\| \leq \|\sum_{n=N+1}^\infty \phi_n \times \psi_n\| < \epsilon$ and $\sum_{n=1}^N \|\phi_n\|_p \|\psi_n\|_{p'} \leq \sum_{n=1}^\infty \|\phi_n\|_p \|\psi_n\|_{p'} < \|\xi\|_p + \epsilon \leq \|\omega\|_p + 2\epsilon$. Thus $\omega \in X_\delta$ and $\|\omega - \xi\|_p < \epsilon$. \square

Theorem 17. Let $1 < p \leq 2$ and $T \in \mathfrak{M}_{p,p}$. Then T extends to a bounded linear map from $\mathcal{A}_p \rightarrow \mathcal{A}_p$, and the linear functional $T^\#: \mathcal{A}_p \rightarrow \mathbb{C}$ given by $T^\#(\omega) = (T\omega)_\zeta$ is in \mathcal{A}_p^* with $\|T^\#\| \leq \|T\|$. Thus $\mathcal{A}_p^* \cong \mathfrak{M}_{p,p}$.

Proof. Let $\delta > 0$ and $\omega \in X_\delta \subset \mathcal{C}_F(\hat{G}) \subset \mathcal{L}^p(\hat{G})$. Write $\omega = \sum_{n=1}^N \phi_n \times \psi_n, \phi_n, \psi_n \in \mathcal{C}_F(\hat{G})$, where $\|\omega\|_p + \delta > \sum_{n=1}^N \|\phi_n\|_p \|\psi_n\|_{p'}$. Now $T\omega = T(\sum_{n=1}^N \phi_n \times \psi_n) = \sum_{n=1}^N T(\phi_n \times \psi_n) = \sum_{n=1}^N (T\phi_n) \times \psi_n$, and $\|T\omega\|_p \leq \sum_{n=1}^N \|T\phi_n\|_p \|\psi_n\|_{p'} \leq \|T\|_{p,p} (\|\omega\|_p + \delta)$. But X_δ is dense in \mathcal{A}_p and so T extends to \mathcal{A}_p with norm less than or equal to $\|T\|_{p,p} (1 + \delta)$. But $\delta > 0$ is arbitrary and so $\|T\omega\|_p \leq \|T\|_{p,p} \|\omega\|_p$. \square

Corollary 18. For $1 \leq r < 2, \mathfrak{M}_{r,r} \subsetneq \bigcap \{\mathfrak{M}_{s,s} : r < s < 2\}$, and for $1 < r \leq 2, \bigcup \{\mathfrak{M}_{s,s} : 1 < s < r\} \subsetneq \mathfrak{M}_{r,r}$.

Proof. J. F. Price [6, pp. 326-330] has given a general argument based on the Riesz-Thorin convexity theorem which yields the corollary using only the facts that $\mathfrak{M}_{q,q} \neq \mathfrak{M}_{2,2}$ ($q < 2$) (see Theorem 9), that $\mathfrak{M}_{q,q}$ is the dual space of \mathcal{A}_q , and that \mathcal{A}_q contains $\mathcal{L}^1(\hat{G})$ as a dense subspace (see Proposition 16). \square

Definition. Let $1 \leq p, q < \infty, 1/p + 1/q \geq 1$, and $1/r = 1/p + 1/q - 1$. We define for $\phi \in \mathcal{L}^r(\hat{G})$,

$$\|\phi\|_{p,q} = \inf$$

The subspace $\mathcal{A}_{p,q}$.

Remark.

$q \leq \infty$, one can prove. (Note

Definition

ω^* denote the $\mathfrak{M}_{p,p}$. Thus $T \phi \in \mathcal{L}^p(\hat{G}), \psi \in \mathcal{A}_p$.

Theorem 1

Proof. Let

$\omega \in \mathcal{A}_p$. Extend that $T^\# \omega = (T_\alpha \omega) \times \psi, S \in \mathfrak{M}_{p,p}$. $T^\#(\phi \times \psi) \xrightarrow{\alpha} T^\# \psi_n \xrightarrow{\alpha} \psi$ in \mathcal{L}^1 . $\phi \times \psi_n \xrightarrow{\alpha} \phi \times \psi = (S\phi) \times \psi$. \square

Corollary 2

Proof. Both

Theorem 2]

$1 < p < \infty$. Then

Proof. Suppose

$T^\#(\omega) = 0$ for all space of $\mathfrak{M}_{p,p}$. $\psi \in \mathcal{L}^p(\hat{G})$. Thus $\omega \in \mathcal{A}_p \subset \mathcal{C}$

Corollary 2

Remark. A)

$$\|\phi\|_{p,q} = \inf \left\{ \sum_{n=1}^{\infty} \|\phi_n\|_p \|\psi_n\|_q : \phi = \sum_{n=1}^{\infty} \phi_n \times \psi_n \text{ (convergence in } \mathcal{L}^r(\hat{G})\text{)}, \right. \\ \left. \{\phi_n\}_{n=1}^{\infty} \subset \mathcal{L}^p(\hat{G}), \{\psi_n\}_{n=1}^{\infty} \subset \mathcal{L}^q(\hat{G}) \right\}.$$

The subspace of $\mathcal{L}^r(\hat{G})$ consisting of all ϕ with $\|\phi\|_{p,q} < \infty$ is denoted by $\mathfrak{A}_{p,q}$.

Remark. For $1 < p < \infty$, observe that $\mathfrak{A}_{p,p'} = \mathfrak{A}_p$; and indeed, for $1 \leq p < q \leq \infty$, one can show that $\mathfrak{A}_{p,q}^* \cong \mathfrak{M}_{p,q}$, by appropriately modifying the preceding proofs. (Note for $p > q$ that $\mathfrak{M}_{p,q} = \{0\}$, and for $1 \leq p < q \leq \infty$ that $1/p + 1/q' > 1$.)

Definition. Let WO denote the weak operator topology on $\mathfrak{M}_{p,p}$, and let w^* denote the weak- $*$ topology on $\mathfrak{M}_{p,p}$ ($1 < p \leq 2$) from the pairing of \mathfrak{A}_p with $\mathfrak{M}_{p,p}$. Thus $T_\alpha \xrightarrow{\alpha} T$ ($\{T_\alpha\}, \{T\} \subset \mathfrak{M}_{p,p}$) in WO if and only if $\langle T_\alpha \phi, \psi \rangle \xrightarrow{\alpha} \langle T \phi, \psi \rangle$, $\phi \in \mathcal{L}^p(\hat{G})$, $\psi \in \mathcal{L}^{p'}(\hat{G})$; and $T_\alpha \xrightarrow{\alpha} T$ in w^* if and only if $T_\alpha^\# \omega \xrightarrow{\alpha} T^\# \omega$, for each $\omega \in \mathfrak{A}_p$.

Theorem 19. In $\mathfrak{M}_{p,p}$ ($1 < p \leq 2$), $WO \subset w^*$.

Proof. Let $T_\alpha, T \in \mathfrak{M}_{p,p}$ with $T_\alpha \xrightarrow{\alpha} T$ in w^* . Thus $T_\alpha^\# \omega \xrightarrow{\alpha} T^\# \omega$ for all $\omega \in \mathfrak{A}_p$. Extend T_α, T to operators from \mathfrak{A}_p to \mathfrak{A}_p (as in Theorem 17) such that $T_\alpha^\# \omega = (T_\alpha \omega)_i$, $T^\# \omega = (T \omega)_i$ ($\omega \in \mathfrak{A}_p$). Let $\phi \in \mathcal{L}^p(\hat{G})$, $\psi \in \mathcal{L}^{p'}(\hat{G})$. We wish to show that $\langle T_\alpha \phi, \psi \rangle \xrightarrow{\alpha} \langle T \phi, \psi \rangle$. It suffices to show that $S(\phi \times \psi) = (S\phi) \times \psi$, $S \in \mathfrak{M}_{p,p}$; for then $\langle T_\alpha \phi, \psi \rangle = ((T_\alpha \phi) \times \psi)_i = (T_\alpha(\phi \times \psi))_i = T_\alpha^\#(\phi \times \psi) \xrightarrow{\alpha} T^\#(\phi \times \psi) = (T(\phi \times \psi))_i = ((T\phi) \times \psi)_i = \langle T\phi, \psi \rangle$. Now let $\psi_n \xrightarrow{n} \psi$ in $\mathcal{L}^{p'}(\hat{G})$, $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{C}_F(\hat{G})$. Then for $S \in \mathfrak{M}_{p,p}$, we have that $\phi \times \psi_n \xrightarrow{n} \phi \times \psi$ in \mathfrak{A}_p and so $S(\phi \times \psi) = \lim_{n \rightarrow \infty} S(\phi \times \psi_n) = \lim_{n \rightarrow \infty} (S\phi) \times \psi_n = (S\phi) \times \psi$. \square

Corollary 20. On bounded subsets of $\mathfrak{M}_{p,p}$ ($1 < p \leq 2$), $w^* = WO$.

Proof. Bounded closed subsets of $\mathfrak{A}_p^* \cong \mathfrak{M}_{p,p}$ are w^* -compact. \square

Theorem 21. Let Φ denote the w^* -closure of $\mathcal{C}_F(\hat{G})$ or $\mathcal{L}^1(\hat{G})$ in $\mathfrak{M}_{p,p}$, $1 < p < \infty$. Then $\Phi = \mathfrak{M}_{p,p}$.

Proof. Suppose $\Phi \neq \mathfrak{M}_{p,p}$, then there exists $\omega \in \mathfrak{A}_p$ such that $\omega \neq 0$ and $T^\#(\omega) = 0$ for all $T \in \mathcal{C}_F(\hat{G}) \subset \mathfrak{M}_{p,p}$. But if $T \in \mathcal{C}_F(\hat{G})$, considered as a subspace of $\mathfrak{M}_{p,p}$, then there exists a $\phi \in \mathcal{C}_F(\hat{G})$ such that $T\psi = \phi \times \psi$ for all $\psi \in \mathcal{L}^p(\hat{G})$. Thus $T^\#(\omega) = (T\omega)_i = (\phi \times \omega)_i = \langle \phi, \omega \rangle = 0$, for all $\phi \in \mathcal{C}_F(\hat{G})$. But $\omega \in \mathfrak{A}_p \subset \mathcal{C}_0(\hat{G})$, so $\omega = 0$. \square

Corollary 22. For $1 < p < \infty$, $\mathcal{C}_F(\hat{G})$ is WO -dense in $\mathfrak{M}_{p,p}$.

Remark. An invariant mean on $\mathcal{L}^\infty(\hat{G})$ is a bounded linear functional p on

$\mathcal{L}^\infty(\hat{G})$ such that (1) $p(\phi) \geq 0$ whenever $\phi \geq 0$, (2) $p(I) = 1$ (I is the identity in $\mathcal{L}^\infty(\hat{G})$), and (3) $p(\hat{f} \times \phi) = f(e)p(\phi)$, $f \in \Lambda(G)$, $\phi \in \mathcal{L}^\infty(\hat{G})$. In [2] we showed that invariant means exist on $\mathcal{L}^\infty(\hat{G})$.

Let p be an invariant mean on $\mathcal{L}^\infty(\hat{G})$. Define $T: \mathcal{L}^\infty(\hat{G}) \rightarrow \mathcal{L}^\infty(\hat{G})$ by $\langle \psi, T\phi \rangle = p(\psi \times \phi)$, $\psi \in \mathcal{L}^1(\hat{G})$, $\phi \in \mathcal{L}^\infty(\hat{G})$; and so $T\phi = p(\phi)I$. Thus $T \in \text{Hom}_{\mathcal{L}^1(\hat{G})}(\mathcal{L}^\infty(\hat{G}), \mathcal{L}^\infty(\hat{G}))$. Also $Tf = 0$ for $f \in L^1(G)^\wedge$, and it follows that T annihilates $\mathcal{C}_0(\hat{G}) = \text{cl}(L^1(G)^\wedge)$ (closure in $\mathcal{C}_0(\hat{G})$): since for $\mu \in M(G)$, $p(\hat{\mu}) = \mu(\{e\})$ (see [3]).

BIBLIOGRAPHY

1. C. Dunkl and D. Ramirez, *Topics in harmonic analysis*, Appleton-Century-Crofts, New York, 1971.
2. ———, *Existence and nonuniqueness of invariant means on $\mathcal{L}^\infty(\hat{G})$* , Proc. Amer. Math. Soc. 32 (1972), 525–530.
3. ———, *Helson sets in compact and locally compact groups*, Michigan Math. J. 19 (1972), 65–69.
4. A. Figà-Talamanca, *Translation invariant operators in L^p* , Duke Math. J. 32 (1965), 495–501. MR 31 #6095.
5. S. Helgason, *Lacunary Fourier series on noncommutative groups*, Proc. Amer. Math. Soc. 9 (1958), 782–790. MR 20 #6667.
6. J. Price, *Some strict inclusions between spaces of L^p -multipliers*, Trans. Amer. Math. Soc. 152 (1970), 321–330.
7. M. Rieffel, *Multipliers and tensor products of L^p -spaces of locally compact groups*, Studia Math. 33 (1969), 71–82. MR 39 #6078.

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