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Subalgebras of the dual of the Fourier algebra of a compact group

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We let G denote an infinite compact group and $\hat{G}$ its dual. We use the notation of our book ((1), Chapters 7 and 8). Recall $A(G)$ denotes the Fourier algebra of G (an algebra of continuous functions on G), and $\mathscr{L}^\infty(\hat{G})$ denotes its dual space under the pairing $\langle f, \phi \rangle$ ($f \in A(G)$, $\phi \in \mathscr{L}^\infty(\hat{G})$). Further, note $\mathscr{L}^\infty(\hat{G})$ is identified with the C^*-algebra of bounded operators on $L^2(G)$ commuting with left translation. The module action of $A(G)$ of $\mathscr{L}^\infty(\hat{G})$ is defined by the following: for $f \in A(G)$, $\phi \in \mathscr{L}^\infty(\hat{G})$, $f.\phi \in \mathscr{L}^\infty(\hat{G})$ by $\langle g, f.\phi \rangle = \langle fg, \phi \rangle$, $g \in A(G)$. Also $\|f.\phi\|_\infty \leqslant \|f\|_A \|\phi\|_\infty$.

Let $\phi \in \mathscr{L}^\infty(\hat{G})$. We call ϕ an almost periodic (weakly almost periodic) functional on $A(G)$ if and only if the map $f \mapsto f.\phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ is a compact (respectively, weakly compact) operator. The space of all such is denoted by $AP(\hat{G})$ (respectively, $W(\hat{G})$).

The object of this paper is to show that $AP(\hat{G})$ and $W(\hat{G})$ are algebras for a restricted class of compact groups called groups of bounded representation type.

Let G be a compact non-Abelian group; we let $\hat{G}$ denote the set of equivalence classes of continuous unitary irreducible representations of $\hat{G}$. We call $\hat{G}$ the dual of G. For $\alpha \in \hat{G}$, let T_α be an element of α. Then T_α is a homomorphism of G into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices, where n_α is the dimension of α. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $x \in G$, $1 \leqslant i,j \leqslant n_\alpha$, and $T_{\alpha ij}$ to denote the function $x \mapsto T_\alpha(x)_{ij}$. Now

$$T_\alpha(xy)_{ij} = \sum_{k=1}^{n_\alpha} T_\alpha(x)_{ik} T_\alpha(y)_{kj} \; (x, y \in G), \quad \text{and} \quad T_\alpha(y^{-1})_{ij} = \overline{T_\alpha(y)_{ji}}.$$

Furthermore, $T_{\alpha ij} \in C(G)$, the space of continuous functions on G.

Let X be an n-dimensional complex inner product space with norm $|\cdot|$. Let $\mathscr{B}(X)$ be the space of linear maps from $X \to X$. We define the operator norm of $A \in \mathscr{B}(X)$ by $\|A\|_\infty = \sup\{|A\xi| : \xi \in X, |\xi| \leqslant 1\}$. The trace of A, $\operatorname{Tr} A$, is $\sum_{i=1}^{n} (A\xi_i, \xi_i)$ where $\{\xi_i\}_{i=1}^n$ is any orthonormal basis for X and $(\cdot, \cdot)$ denotes the inner product in X. Let $|A|$ denote $(A^*A)^{\frac{1}{2}}$. The value $\|A\|_\infty$ is the spectral radius of $|A|$; that is, $\max\{\lambda_i : 1 \leqslant i \leqslant n\}$, where λ_i are the eigenvalues of $|A|$.

Let ϕ be a set $\{\phi_\alpha : \alpha \in \hat{G}$ where $\phi_\alpha \in \mathscr{B}(C^{n_\alpha})\}$ such that the supremum of $\{\|\phi_\alpha\|_\infty : \alpha \in \hat{G}\}$ is finite. The set of all such ϕ is denoted by $\mathscr{L}^\infty(\hat{G})$. It is a Banach algebra under the norm $\|\phi\|_\infty = \sup\{\|\phi_\alpha\|_\infty : \alpha \in \hat{G}\}$ and coordinate-wise operations.

Let $M(G)$ denote the measure algebra of G; that is, the space of finite regular Borel measures on G with convolution as multiplication. For $\mu \in M(G)$, the Fourier–Stieltjes transform of μ, $\hat{\mu}$, is a matrix-valued function defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1})\, d\mu(x).$$

Note $\hat{\mu} \in \mathscr{L}^\infty(\hat{G})$. Indeed, $\|\hat{\mu}\|_\infty \leqslant \|\mu\|$.

Let $A \in \mathscr{B}(X)$ where X is a finite-dimensional complex inner product space. We define the dual norm to $\|\cdot\|_\infty$ by $\|A\|_1 = \sup\{|\mathrm{Tr}\,(AB)| : \|B\|_\infty \leqslant 1\}$. This norm can be also characterized by $\|A\|_1 = \mathrm{Tr}\,(|A|)$. For $\phi \in \mathscr{L}^\infty(\hat{G})$, we put $\|\phi\|_1 = \Sigma_{\alpha\in\hat{G}} n_\alpha \|\phi_\alpha\|_1$. Let $\mathscr{L}^1(\hat{G}) = \{\phi \in \mathscr{L}^\infty(\hat{G}) : \|\phi\|_1 < \infty\}$. The space $\mathscr{L}^1(\hat{G})$ is a Banach space under the norm $\|\cdot\|_1$. For $\phi \in \mathscr{L}^1(\hat{G})$, let $\mathrm{Tr}\,(\phi) = \Sigma_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}\,(\phi_\alpha)$.

We will now define $A(G)$, the Fourier algebra of G. Let $A(G)$ be the set of $f \in C(G)$ for which $\hat{f} \in \mathscr{L}^1(\hat{G})$. We norm $A(G)$ by

$$\|f\|_A = \|\hat{f}\|_1 = \Sigma_{\alpha\in\hat{G}} n_\alpha \|\hat{f}_\alpha\|_1 < \infty.$$

Note that $A(G)$ is isomorphic to $\mathscr{L}^1(\hat{G})$ by $f \mapsto \hat{f}$ because for any $\phi \in \mathscr{L}^1(\hat{G})$ the function $f(x) = \Sigma_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}\,(\phi_\alpha T_\alpha(x))$ is in $A(G)$; further,

$$\|f\|_\infty = \sup\{|\Sigma_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}\,(\phi_\alpha T_\alpha(x))| : x \in G\} \leqslant \Sigma_{\alpha\in\hat{G}} n_\alpha \|\phi_\alpha\|_1 = \|\phi\|_1.$$

We now recall the following facts (see (1), Chapter 8): $A(G)$ is an algebra under pointwise multiplication; the dual of $\mathscr{L}^1(\hat{G})$ is $\mathscr{L}^\infty(\hat{G})$ and the correspondence is given by $\langle \Psi, \phi\rangle = \mathrm{Tr}\,(\phi\Psi)$, $\Psi \in \mathscr{L}^1(\hat{G})$, $\phi \in \mathscr{L}^\infty(\hat{G})$. Thus the dual of $A(G)$ can be identified with $\mathscr{L}^\infty(\hat{G})$ and the pairing is given by $\langle f, \phi\rangle = \mathrm{Tr}\,(\phi\hat{f})$, $f \in A(G)$, $\phi \in \mathscr{L}^\infty(\hat{G})$. The module action of $\mathscr{L}^\infty(\hat{G})$ over $A(G)$ is defined by the following: $f \in A(G)$, $\phi \in \mathscr{L}^\infty(\hat{G})$, $f.\phi \in \mathscr{L}^\infty(\hat{G})$ by $\langle g, f.\phi\rangle = \langle fg, \phi\rangle$, $g \in A(G)$. Note $\|f.\phi\|_\infty \leqslant \|f\|_A \|\phi\|_\infty$.

Let $L(x)$ be the left translation operator given by $L(x)f(y) = f(x^{-1}y)$, $f \in C(G)$, $x, y \in G$. Then for $f \in L^1(G)$,

$$(L(x)f)\hat{}_\alpha = \hat{f}_\alpha T_\alpha(x^{-1}) \quad (x \in G,\ \alpha \in \hat{G}).$$

For $f \in A(G)$ and $\phi \in \mathscr{L}^\infty(\hat{G})$, $\phi\hat{f} \in \mathscr{L}^1(\hat{G})$.

We thus define $\phi f \in A(G)$ by

$$\phi f(x) = \Sigma_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}\,(\phi_\alpha \hat{f}_\alpha T_\alpha(x)) \quad (x \in G).$$

Further, $L(x)(\phi f) = \phi(L(x)f)$ $(x \in G)$. Thus for $f \in A(G)$ and $\phi \in \mathscr{L}^\infty(\hat{G})$, we have that

$$\phi f(x) = L(x^{-1})\,\phi f(e) = \phi L(x^{-1}) f(e) = \langle L(x^{-1})f, \phi\rangle \quad (x \in G).$$

Also $(\phi f)\hat{}_\alpha = \phi_\alpha \hat{f}_\alpha$ $(\alpha \in \hat{G})$.

Let $\alpha \in \hat{G}$, then

$$L(x)\, T_\alpha(y)_{ij} = T_\alpha(x^{-1}y)_{ij} = \sum_{k=1}^{n_\alpha} T_\alpha(x^{-1})_{ik} T_\alpha(y)_{kj} \quad (x, y \in G).$$

Thus

$$L(x)\, T_{\alpha ij} = \sum_{k=1}^{n_\alpha} T_\alpha(x^{-1})_{ik} T_{\alpha kj} \quad (1 \leqslant i, j \leqslant n_\alpha).$$

Now let $f \in A(G)$ and ϕ, $\Psi \in \mathscr{L}^\infty(\hat{G})$, then

$$\langle f, \phi\Psi\rangle = \Sigma_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}\,(\phi_\alpha \Psi_\alpha \hat{f}_\alpha) = \Sigma_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}\,(\Psi_\alpha \hat{f}_\alpha \phi_\alpha) = \langle \Psi f, \phi\rangle.$$

Now let x be fixed in $G, f \in A(G)$, $\Psi \in \mathscr{L}^\infty(\hat{G})$, and $\alpha \in G$. Then

$$\begin{aligned}\Psi(T_{\alpha ij}f)(x) &= \langle L(x^{-1})(T_{\alpha ij}f), \Psi\rangle \\ &= \langle (L(x^{-1})T_{\alpha ij})(L(x^{-1})f), \Psi\rangle \\ &= \left\langle \sum_{k=1}^{n_\alpha} T_\alpha(x)_{ik}\, T_{\alpha kj}\, L(x^{-1})f, \Psi \right\rangle \\ &= \sum_{k=1}^{n_\alpha} T_\alpha(x)_{ik} \langle L(x^{-1})f, T_{\alpha kj}.\Psi\rangle \\ &= \sum_{k=1}^{n_\alpha} T_\alpha(x)_{ik}\, ((T_{\alpha kj}.\Psi)f(x)).\end{aligned}$$

Thus

$$\Psi(T_{\alpha ij}f) = \sum_{k=1}^{n_\alpha} T_{\alpha ik}((T_{\alpha kj}.\Psi)f).$$

THEOREM 1. *Let $\phi, \Psi \in \mathscr{L}^\infty(\hat{G})$. Then $T_{\alpha ij}.(\phi\Psi) = \sum_{k=1}^{n_\alpha}(T_{\alpha ik}.\phi)(T_{\alpha kj}.\Psi)$, for $\alpha \in \hat{G}$ and $1 \leqslant i,j \leqslant n_\alpha$.*

Proof. Let $f \in A(G)$, then

$$\begin{aligned}\langle f, T_{\alpha ij}.(\phi\Psi)\rangle &= \langle T_{\alpha ij}f, \phi\Psi\rangle \\ &= \langle \Psi(T_{\alpha ij}f), \phi\rangle = \left\langle \sum_{k=1}^{n_\alpha} T_{\alpha ik}((T_{\alpha kj}.\Psi)f), \phi \right\rangle \\ &= \sum_{k=1}^{n_\alpha} \langle (T_{\alpha kj}.\Psi)f, T_{\alpha ik}.\phi\rangle \\ &= \sum_{k=1}^{n_\alpha} \langle f, (T_{\alpha ik}.\phi)(T_{\alpha kj}.\Psi)\rangle \\ &= \left\langle f, \sum_{k=1}^{n_\alpha}(T_{\alpha ik}.\phi)(T_{\alpha kj}.\Psi)\right\rangle. \quad \square\end{aligned}$$

In a series of papers, we have been studying the non-Abelian extension (to $\hat{G}$) of the space of almost periodic (weakly almost periodic) functions. We say for $\phi \in \mathscr{L}^\infty(\hat{G})$ that ϕ is almost periodic (weakly almost periodic) if and only if the map $f \mapsto f.\phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ is a compact (weakly compact) operator. The space of all such ϕ is denoted by $AP(\hat{G})$ (respectively, $W(\hat{G})$). We showed in (2) that both $AP(G)$ and $W(\hat{G})$ are closed $*$-subspaces of $\mathscr{L}^\infty(\hat{G})$ ($*$ denotes the adjoint operation). Each is a module over $A(G)$, and each possesses a system of almost invariant integrals which defines an (unique) invariant mean.

We study in this paper a restricted class of compact groups for which we can show that $AP(\hat{G})$ and $W(\hat{G})$ are algebras. One says that the compact group G is of bounded representation type if $\sup\{n_\alpha : \alpha \in \hat{G}\}$ is finite. These compact groups can be characterized as extensions of abelian compact groups by finite groups. This characterization is due to Calvin Moore.

THEOREM 2. *Let G be an infinite compact group of bounded representation type. Then the space $AP(\hat{G})$ is an algebra.*

Proof. Let B denote the unit ball in $A(G)$. Under the hypothesis that G is of bounded representation type, the closed balanced convex hull of $\{T_{\alpha ij} : \alpha \in \hat{G},\ 1 \leqslant i, j \leqslant n_\alpha\}$ in

$A(G)$ contains a nonzero multiple of B. Let ϕ, $\Psi \in AP(\hat{G})$; we wish to show $\{f.(\phi\Psi), f \in B\}$ is relatively compact in $\mathscr{L}^\infty(\hat{G})$. Thus it suffices to show $\{T_{\alpha ij}.(\phi\Psi), \alpha \in \hat{G}, 1 \leqslant i, j \leqslant n_\alpha\}$ is relatively compact in $\mathscr{L}^\infty(\hat{G})$.

Now $S_1 = \{T_{\alpha ij}.\phi : \alpha \in \hat{G}, 1 \leqslant i,j \leqslant n_\alpha\}$ and $S_2 = \{T_{\alpha ij}.\Psi : \alpha \in \hat{G}, 1 \leqslant i,j \leqslant n_\alpha\}$ are relatively compact in $\mathscr{L}^\infty(\hat{G})$, and thus by Theorem 1,

$$T_{\alpha ij}.(\phi\Psi) \in S_1 S_2 + S_1 S_2 + \ldots + S_1 S_2 \ldots \ (\leqslant \sup n_\alpha \text{ times});$$

and thus $\{T_{\alpha ij}.(\phi\Psi)\}$ is relatively compact. Hence $\phi\Psi \in AP(\hat{G})$. □

We wish now to show the analogous result for $W(\hat{G})$. An inspection of the proof of Theorem 2 shows that we have used two facts about the topology of $\mathscr{L}^\infty(\hat{G})$: (1) the closed balanced convex hull of a compact set is compact and (2) the product of two compact sets is compact. The first result is the Mazur theorem ((3), p. 416) and the second result follows from the joint continuity of multiplication in $\mathscr{L}^\infty(\hat{G})$. When one considers $\mathscr{L}^\infty(\hat{G})$ with the weak topology, the first result still holds; and, in fact, it is the Krein–Šmulian theorem ((3), p. 434).

We now study whether the multiplication of two compact sets is a compact set with the weak topology on $\mathscr{L}^\infty(\hat{G})$. We first recall the Eberlein theorem ((3), p. 430), which says that weak compactness is equivalent to weak sequential compactness. The sequences in $\mathscr{L}^\infty(\hat{G})$ which converge weakly have been characterized in (2).

Let $\{\phi_\lambda\}$ be a net in $\mathscr{L}^\infty(\hat{G})$. One says that $\phi_\lambda \overset{\lambda}{\to} \phi \in \mathscr{L}^\infty(\hat{G})$, quasi-uniformly on $\hat{G}$, if and only if $(\phi_\lambda)_\alpha \overset{\lambda}{\to} (\phi)_\alpha$ for each $\alpha \in \hat{G}$ and for each $\epsilon > 0$ and λ_0, there exists a finite number of indices $\lambda_1, \ldots, \lambda_n \geqslant \lambda_0$ such that for each $\alpha \in \hat{G}$,

$$\min\{\|(\phi_{\lambda_i})_\alpha - \phi_\alpha\|_\infty : 1 \leqslant i \leqslant n_\alpha\} < \epsilon.$$

One of our results from (2) is the following fact: let $\{\phi_n\}$ be a sequence in $\mathscr{L}^\infty(\hat{G})$, then $\phi_n \overset{n}{\to} \phi$ ($\phi \in \mathscr{L}^\infty(\hat{G})$) weakly if and only if $\sup\|\phi_n\|_\infty < \infty$ and every subsequence of $\{\phi_n\}$ converges quasi-uniformly on $\hat{G}$ to ϕ.

THEOREM 3. *The product of two weakly compact subsets of $\mathscr{L}^\infty(\hat{G})$ is a weakly compact subset of $\mathscr{L}^\infty(\hat{G})$.*

Proof. Let A, B be weakly compact subsets of $\mathscr{L}^\infty(\hat{G})$. Let $\{\upsilon^{(n)}\}_{n=1}^\infty \subset AB$. Write $\upsilon^{(n)} = \phi^{(n)}\psi^{(n)}$, $\phi^{(n)} \in A$, $\psi^{(n)} \in B$. Now there are ϕ, ψ in A, B respectively such that (by passing to a subsequence) $\phi^{(n)} \overset{n}{\to} \phi$, $\psi^{(n)} \to \psi$ weakly in $\mathscr{L}^\infty(\hat{G})$. We may assume that $\phi = 0 = \psi$. (Here we are using the fact that multiplication in $\mathscr{L}^\infty(\hat{G})$ with the weak topology is separately continuous.) We wish to show that $\upsilon^{(n)} \overset{n}{\to} 0$ weakly. Since A and B are bounded subsets of $\mathscr{L}^\infty(\hat{G})$, the sequence $\{\upsilon^{(n)}\}_{n=1}^\infty$ is bounded. Let $\{\upsilon^{(n_k)}\}_{k=1}^\infty$ be a subsequence of $\{\upsilon^{(n)}\}_{n=1}^\infty$. Let $\epsilon > 0$ and k_0 a positive integer. There exists a finite set of positive integers $k_1, k_2, \ldots, k_m \geqslant k_0$ such that for each $\alpha \in \hat{G}$,

$$\min\{\|\phi_\alpha^{(k_i)}\|_\infty : 1 \leqslant i \leqslant m\} < \epsilon^{\frac{1}{2}} \quad \text{and} \quad \min\{\|\psi_\alpha^{(k_i)}\|_\infty : 1 \leqslant i \leqslant m\} < \epsilon^{\frac{1}{2}}.$$

Thus for each $\alpha \in \hat{G}$, $\min\{\|(\phi\psi)_\alpha^{(k_i)}\|_\infty : 1 \leqslant i \leqslant m\} < \epsilon$. Finally, since $(\phi^{(n_k)})_\alpha \overset{k}{\to} (\phi)_\alpha$ and $(\psi^{(n_k)})_\alpha \overset{k}{\to} \psi_\alpha$ for each $\alpha \in \hat{G}$, $((\phi\psi)^{(n_k)})_\alpha \overset{k}{\to} (\phi\psi)_\alpha$ for each $\alpha \in \hat{G}$. Thus $\{\upsilon^{(n)}\}_{n=1}^\infty$ has a weak cluster point, and so AB is weakly compact in $\mathscr{L}^\infty(\hat{G})$. □

We now have the analogue of Theorem 2 for $W(\hat{G})$.

THEOREM 4. *Let G be an infinite compact group of bounded representation type. Then the space $W(\hat{G})$ is an algebra.*

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