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For a compact group $G$ with dual $\hat{G}$, a set $F \subseteq \hat{G}$ is a Sidon set if and only if any unitary operator valued function on $E$ can be interpolated almost surely by a Fourier-Stieltjes transform. Further, $E$ is an $A_p$ set for $p > 2$ if and only if any unitary operator valued function on $E$ can be interpolated by an element of the multiplier algebra of $L^p(G)$. (Received January 10, 1972.)
Characterizations of Sidon Sets
and $\Lambda_p$ Sets on Compact Groups

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Abstract

For a compact group $\mathcal{G}$ with dual $\hat{\mathcal{G}}$, a set $E \subset \hat{\mathcal{G}}$ is a Sidon set if and only if any unitary operator valued function on $E$ can be interpolated almost surely by a Fourier-Stieltjes transform. Further, $E$ is a $\Lambda_p$ set for $p > 2$ if and only if any unitary operator valued function on $E$ can be interpolated by an element of the multiplier algebra of $L^p(\mathcal{G})$. 
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Let $G$ be an infinite compact group and $\hat{G}$ its dual (we will use the notation of [1, Chapters 7 and 8]). For $\alpha \in \hat{G}$, let $T_\alpha \in \alpha$. Then $T_\alpha$ is a continuous homomorphism of $G$ into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices. We use $T_\alpha(i, j)$ to denote the matrix entries of $T_\alpha(x)$, $1 \leq i, j \leq n_\alpha$. Let $\chi_\alpha(x) = Tr(T_\alpha(x))$ ($\text{Tr}$ = trace). We call $\chi_\alpha$ the character of $\alpha$. Let $\phi$ be a set $\{\phi_\alpha : \alpha \in \hat{G}\}$ where $\phi_\alpha \in B(C_0^n)$ be such that sup $\{\|\phi_\alpha\|_\infty : \alpha \in \hat{G}\} < \infty$ where $\|\cdot\|_\infty$ denotes the operator norm. The set of all such $\phi$ is denoted by $\mathcal{K}^\infty(\hat{G})$. It is a Banach algebra under the norm $\|\phi\|_\infty = \sup \{\|\phi_\alpha\|_\infty : \alpha \in \hat{G}\}$ and coordinatewise operations. For $\mu \in M(G)$, the Fourier-Stieltjes transform of $\mu$, $\hat{\mu}$, is a matrix-valued function defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto \hat{\mu}_\alpha = \int_{G} T_\alpha(x^{-1}) \, d\mu(x).$$

Note that $\hat{\mu} \in \mathcal{K}^\infty(\hat{G})$. We denote the center of $M(G)$ by $Z M(G)$. Note that for $\mu \in M(G)$, $\mu \in Z M(G)$ if and only if $\hat{\mu}_\alpha = c_\alpha I_{n_\alpha}$, when $I_{n_\alpha}$ denotes the identity operator on $C_0^n$.

Let $E \subseteq \hat{G}$. We say that $E$ is a Sidon set if and only if given any $\phi \in \mathcal{K}^\infty(\hat{G})$, there is $\mu \in M(G)$ such that $\hat{\mu}_\alpha = \phi_\alpha$ for $\alpha \in E$.

We say that $E$ is a central Sidon set if and only if given any $\phi \in Z \mathcal{K}^\infty(\hat{G})$, (the center of $\mathcal{K}^\infty(\hat{G})$) there is $\mu \in Z M(G)$ such that $\hat{\mu}_\alpha = \phi_\alpha$ for $\alpha \in E$. 
Following Rudin [6, p. 121] one can show that \( E \subset \hat{G} \) is a Sidon set if and only if given any \( \phi \in X^\infty(\hat{G}) \) where \( \hat{\phi}_\alpha \) is unitary and hermitian for \( \alpha \in E \), there is \( \mu \in M(G) \) such that \( \sup_{\alpha \in E} \{ \| \hat{\phi}_\alpha - \mu\hat{\eta}_\alpha \|_\infty : \} < 1 \). Also \( E \subset G \) is a central Sidon set if and only if given any \( \phi \in \mathcal{D}X^\infty(\hat{G}) \) where \( \hat{\phi}_\alpha \) is unitary and hermitian (and thus \( \hat{\phi}_\alpha = \pm I^n \)), there is \( \mu \in M(G) \) such that \( \sup_{\alpha \in E} \{ \| \hat{\phi}_\alpha - \mu\hat{\eta}_\alpha \|_\infty : \alpha \in E \} < 1 \) (for example, see [4, p. 448] or [5]).

For \( E \subset \hat{G} \), we let \( \Omega(E) = \prod_{\alpha \in E} U(n_\alpha) \) with its natural topology and probability measure. We now are able to state precisely our first result.

**Theorem 1:** Let \( G \) be an infinite compact group and \( E \subset \hat{G} \). For \( E \) to be a Sidon set it is necessary and sufficient that there is a Borel set of positive measure \( S \subset \Omega(E) \) such that for \( \omega \in S \), there is a \( \mu \in M(G) \) such that \( \hat{\mu}|E = \omega \).

**Proof:** If \( E \) is a Sidon set, then we can interpolate always and so the condition is satisfied. Now suppose we can interpolate with a positive probability.

Let \( B \) be the unit ball in \( M(G) \). Now \( B \) is weak-* compact. The topology we need to consider on \( M(G) \) is the topology \( \mathcal{T} \) of pointwise convergence on \( \hat{G} \) of the Fourier-Stieltjes transforms. Now since \( G \) is compact, the weak-* topology is equivalent to \( \mathcal{T} \) on \( B \). It follows that \( M(G) \) is \( \sigma \)-compact in \( \mathcal{T} \) and that \( \Lambda = (M(G)|E) \cap \Omega \) is a measurable subset of \( \Omega \). Since \( \Lambda \) is a subgroup with positive measure it is open (by the Steinhaus theorem). But open subgroups are closed and since \( \Lambda \) is dense in \( \Omega \) it is all of \( \Omega \). We are done now by the previous quoted result of Rudin. \( \square \)
Remark: The zero-one law guarantees us that \((M(G)^\hat{\cdot} \mid E) \cap \Omega\) has measure either zero or one.

We can use the method of proof to arrive at other similar results. We state one now as an example.

Corollary 2: Let \(G\) be an infinite compact group and \(E \subseteq \hat{G}\). For \(E\) to be a central Sidon set it is necessary and sufficient that there is a Borel set of positive measure \(S \subseteq \Omega = \Pi_{\alpha \in E} \{1, \omega\}\) such that for \(\omega \in S\), there is a \(\mu \in M(G)\) such that \(\hat{\mu} \mid E = \omega\).

Let \(E \subseteq \hat{G}\). We say that \(E\) is a \(\Lambda_p\) set \((1 \leq p < \infty)\) if and only if \(M_p(G) = \{\mu \in M(G) : \hat{\mu}_\alpha = 0\text{ for }\alpha \notin E\}\) is setwise equal to \(L^p_E(G) = \{f \in L^p(G) : \hat{f}_\alpha = 0\text{ for all }\alpha \notin E\}\), (see [4, p. 420]).

Let \(M_p\) denote the subset of \(\mathcal{S}(\hat{G})\) such that \(\hat{f} \in L^p(G)\) for each \(f \in L^p(G)\) where \(\hat{f}\) is defined by the Fourier series \(\sum_{\alpha \in \hat{G}} \hat{f}_\alpha \cdot \hat{\alpha}\). The set \(M_p\) is a subalgebra of \(\mathcal{S}(\hat{G})\) and is called the \(L^p\) multiplier algebra (see [2]). We denote the set \(\{\phi \in M_p : \hat{\phi}_\alpha = 0\text{ for }\alpha \notin E\} = M_p^E\).

We now observe that \(\Lambda_p\) sets have a characterization similar to Sidon sets.

Theorem 3: For \(p > 2\), \(E \subseteq \hat{G}\) is a \(\Lambda_p\) set if and only if for each \(\omega \in \Omega(E) = \Pi_{\alpha \in \hat{G}} U(\alpha)\), there is a \(\psi \in M_p^E\) such that \(\hat{\psi} \mid E = \omega\).

Proof: Let \(E\) be a \(\Lambda_p\) set \((p > 2)\), and let \(\omega \in \Omega(E)\). Given \(f \in L^p(G)\), we need to show that \(\hat{\omega} f \in L^p_E(G) = L^2_E(G)\) (see [4, p. 420]). But \(\|\hat{\omega} f\|_2 = \|\hat{\omega} f\|_2 \leq \|\omega\|_\infty \|f\|_2 = \|\omega\|_\infty \|f\|_2 < \infty\).

Now let \(\Omega(E) = M_p^E\). To show that \(E\) is a \(\Lambda_p\) set, it will suffice to show that \(L^2_E(G) \subseteq L^p_E(G)\). Let \(f \in L^2_E(G)\). There is an \(\omega \in \Omega(E)\)
such that \( \hat{\omega} f \in L^p_E(G) \) (see [3]); and thus since \( \omega^* \) is also in \( \Omega(E) = M^p_E \), we have that \( f = \hat{\omega}^* (\hat{\omega} f) \in L^p_E(G) \). □

We now prove a partial result like our previous probabilistic ones.

Theorem 4: Let \( G \) be a compact abelian group and let \( E \subset \hat{G} \). For \( E \) to be a \( \Lambda^p \) set \( (2 < p < \infty) \) it is necessary and sufficient that there is a Borel set of positive measure \( S \subset \Omega(E) \) such that for \( \omega \in S \), there is a \( \phi \in M^p_E \) such that \( \phi|E = \omega \).

Proof: We first observe that \( M^p_E \cap \Omega(E) \) is measurable. This follows as before since \( M^p \) is a dual space (see [2]). Now repeat the proof of Theorem 1. □

We are not able to handle the nonabelian compact case since it does not seem to be known whether \( M^p \) is closed under the adjoint operation of \( \mathcal{X}^\infty(\hat{G}) \). However, we could define central \( \Lambda^p \) sets (see [5]) for which the result analogous to Corollary 2 holds.
Bibliography


Footnotes

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