

BOUNDED PROJECTIONS ON FOURIER-STIELTJES TRANSFORMS

CHARLES F. DUNKL AND DONALD E. RAMIREZ¹

ABSTRACT. We study certain algebraic projections on the measure algebra (of a locally compact abelian group) which extend to bounded projections on the uniform closure of the Fourier-Stieltjes transforms. These projections arise by studying a Raikov system of subsets induced by locally compact subgroups. These results generalize the inequality $\|\hat{\mu}_d\|_\infty \leq \|\hat{\mu}\|_\infty$ (where μ is in the measure algebra, μ_d is the discrete part of μ , and $\|\hat{\mu}\|_\infty$ is the sup-norm of the Fourier-Stieltjes transform).

Here H will be a locally compact abelian (LCA) group. The group H with the discrete topology is denoted H_d . This is the same as giving H the topology induced from declaring the subgroup $G = \{0\} \subset H$ to be open. The space of finite regular Borel measures on H is denoted $M(H)$. For $\mu \in M(H)$, let μ_d denote the discrete part of μ . The ring homomorphism $\mu \mapsto \mu_d$ maps $M(H)$ onto $M(H_d)$, and this map is norm-nonincreasing in the measure norm; that is, $\|\mu_d\| \leq \|\mu\|$, $\mu \in M(H)$. For $\mu \in M(H)$, we let $\hat{\mu}$ denote the Fourier-Stieltjes transform of μ ; that is

$$\hat{\mu}(\gamma) = \int_H (\gamma(x))^{-1} d\mu(x),$$

$\gamma \in \hat{H}$ (the dual of H). In two previous papers [2], [3], we showed (in a more general setting) $\|\hat{\mu}_d\|_\infty \leq \|\hat{\mu}\|_\infty$, $\mu \in M(H)$ (where $\|\cdot\|_\infty$ denotes the sup-norm). This further implies that $\mathcal{M}(\hat{H}) = \mathcal{M}_c(\hat{H}) \oplus \mathcal{M}_d(\hat{H})$, where $\mathcal{M}(\hat{H})$, $\mathcal{M}_c(\hat{H})$, and $\mathcal{M}_d(\hat{H})$ are the sup-norm closures on \hat{H} of the Fourier-Stieltjes transforms of measures from $M(H)$, $M_c(H)$ (the space of continuous measures), and $M(H_d)$ respectively. Let Δ denote the maximal ideal space of $M(H)$, and let $\kappa\hat{H}$ denote the Δ -closure of \hat{H} in Δ . (Recall $\hat{H} \subset \Delta$ under the identification map from \hat{H} to Δ by $\pi_\gamma(\mu) = \hat{\mu}(\gamma)$, $\gamma \in \hat{H}$, $\mu \in M(H)$.) We call the set $\kappa\hat{H} \setminus \hat{H}$ the fringe of \hat{H} . The result

Received by the editors February 8, 1971.

AMS 1970 subject classifications. Primary 43A10, 43A25.

Key words and phrases. Measure algebra of a locally compact abelian group, Fourier-Stieltjes transform, Raikov system of subsets, positive definite function.

¹ This research was supported in part by NSF contract number GP-19852.

© American Mathematical Society 1972

$\|\hat{\mu}_d\|_\infty \leq \|\hat{\mu}\|_\infty$ ($\mu \in M(H)$) implies that the fringe of \hat{H} contains a homeomorphic copy of the Bohr group $\beta\hat{H}$ of \hat{H} (under the map $\chi \mapsto \pi_\chi$ from $\beta\hat{H}$ to Δ given by $\pi_\chi(\mu) = \int_H \tilde{\chi} d\mu_d$, $\mu \in M(G)$, $\chi \in \beta\hat{H}$).

The setting in this paper is as follows. We let H be an LCA group with topology \mathfrak{C}_H , and G a subgroup of H which has an LCA group topology \mathfrak{C}_G such that the injection $(G, \mathfrak{C}_G) \rightarrow (H, \mathfrak{C}_H)$ is continuous. For example, suppose G is the image under a continuous monomorphism of an LCA group. We let H_G denote H with the topology induced by declaring the subgroup G with the \mathfrak{C}_G -topology to be open. We will assume that G is a *nonopen* subgroup of H so that $H \neq H_G$ topologically.

We now will define the natural projection $P: M(H) \rightarrow M(H_G)$ by utilizing a Raikov system of subsets of H . (For the basic facts concerning Raikov systems see [5].) Let \mathcal{F} denote a family of σ -compact subsets of H such that: (1) if $A \in \mathcal{F}$, B is σ -compact, and $B \subset A$, then $B \in \mathcal{F}$, (2) if $\{A_n\}_{n=1}^\infty \subset \mathcal{F}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$, (3) if $A, B \in \mathcal{F}$, then $A + B \in \mathcal{F}$, and (4) if $A \in \mathcal{F}$ and $x \in H$, then $x + A \in \mathcal{F}$. Such a family of subsets of H is called a Raikov system. We choose \mathcal{F} to be the Raikov system generated by the family of compact subsets of G .

Let R be the set of measures $\mu \in M(H)$ such that $|\mu|$ is concentrated on some elements of \mathcal{F} , and let I be the set of measures $\mu \in M(H)$ such that $|\mu|(A) = 0$ for all $A \in \mathcal{F}$. Then I is a closed ideal in $M(H)$ and R is a closed subalgebra of $M(H)$. Furthermore, $M(H) = R \oplus I$ (see, for example, [5, p. 151]). Now R can be identified with $M(H_G)$, and thus the natural projection $P: M(H) \rightarrow M(H_G)$ is induced by the given direct sum. For $\mu \in M(H)$, we write $\mu = \mu_G + \mu_I$ where $\mu_G \in M(H_G)$ and $\mu_I \in I$. Thus $P\mu = \mu_G$, $\mu \in M(H)$. Observe that P is a norm-bounded projection; that is, $\|P\mu\| \leq \|\mu\|$, $\mu \in M(H)$. Our goal now is to show

$$\|(P\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty, \quad \mu \in M(H).$$

Let $\phi: H_G \rightarrow H$ be the identity map and $\hat{\phi}: \hat{H} \rightarrow \hat{H}_G$ the adjoint map (an injection). In an earlier paper [4], we showed for any continuous homomorphism $\pi: G_1 \rightarrow G_2$ (G_1, G_2 LCA groups) that π is open if and only if $\hat{\pi}: \hat{G}_2 \rightarrow \hat{G}_1$ (the adjoint map) is proper (the inverse image of a compact set is compact). Thus since ϕ is not open, $\hat{\phi}$ is not proper. The map ϕ induces a continuous homomorphism $\phi^*: M(H_G) \rightarrow M(H)$. Since ϕ is one-to-one, $\hat{\phi}\hat{H}$ is dense in \hat{H}_G . Indeed for any compact $K \subset \hat{H}$, $\hat{\phi}(\hat{H} \setminus K)$ is dense in \hat{H}_G . For $\mu \in M(H_G)$, $\|\hat{\mu}\|_\infty$ is the supremum of $|\hat{\mu}|$ over either $\hat{\phi}\hat{H}$ or \hat{H}_G . (We will identify $\hat{\phi}\hat{H}$ and \hat{H} as subsets of \hat{H}_G when convenient.)

For an LCA group L , we let $P(L)$ denote the space of continuous positive definite functions on L ; we let $P_c(L)$ be those $f \in P(L)$ with compact support.

We will denote the Haar measure on H_G by λ . (The measure λ restricted to G is the Haar measure on G .)

PROPOSITION 1. *Let $f \in P_c(H_G)$ and let $d\mu = f d\lambda$. If $g \in P_c(H)$, then $g * \mu$ (convolution in $M(H)$) is in $P_c(H)$.*

PROOF. Since $f \in P_c(H_G)$, $f \in L^1(\hat{H}_G)$ by the inversion theorem [7, p. 22], and $f \geq 0$ by Bochner's theorem [7, p. 19]. Thus for $\gamma \in \hat{H} \subset \hat{H}_G$, $\hat{\mu}(\gamma) = \int_H \bar{\gamma} d\mu = \int_{H_G} \bar{\gamma} f d\lambda = \hat{f}(\gamma) \geq 0$.

Since g and μ have compact supports, $g * \mu$ is a continuous function on H with compact support. Finally, $g * \mu$ is positive definite since $(g * \mu)^\wedge = \hat{g}\hat{\mu} \geq 0$ on \hat{H} . \square

An LCA group L is amenable, and thus satisfies the condition of Godement: the constant function 1 can be approximated uniformly on compact subsets of L by functions of the form $k * \bar{k}$, where k is a continuous function with compact support and $\bar{k}(x) = (k(-x))^-$, $x \in L$. (See [6, p. 168, 172].) Thus we have:

PROPOSITION 2. *Let L be an LCA group and $K \subset L$ a compact subset of L . Given $\varepsilon > 0$, there is $p \in P_c(L)$ such that $p(0) = 1$ and $|p - 1| < \varepsilon$ on K .*

PROPOSITION 3. *Let K be a compact subset of H_G , and let U be a relatively compact neighborhood of 0 in H_G . Then there is a neighborhood V of 0 in H such that $(x + V) \cap K \subset x + U$ for all $x \in K$.*

PROOF. Since K is compact in H_G , $K - K$ is also compact in H_G ; and the induced topology on $K - K$ from H agrees with the H_G -topology on $K - K$ (since compact topologies are minimal Hausdorff). Thus there is an H -open neighborhood of 0, V , such that $V \cap (K - K) \subset U \cap (K - K)$. Thus for $x \in K$, $(x + V) \cap K \subset x + (V \cap (K - \{x\})) \subset x + (V \cap (K - K)) \subset x + (U \cap (K - K)) \subset x + U$. \square

PROPOSITION 4. *Let $\xi \in \hat{H}_G$, K a compact subset of H_G , and $\varepsilon > 0$ be given. Then there exists $\gamma \in \hat{H}$ such that $|\gamma - \xi| < \varepsilon$ on K .*

PROOF. Recall that $\hat{\phi}\hat{H}$ can be identified with \hat{H} , and it is dense in \hat{H}_G . Finally, the topology in \hat{H}_G is the compact-open topology. \square

THEOREM 5. *Let $P: M(H) \rightarrow M(H_G)$. Then $\|(P\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$, $\mu \in M(H)$.*

PROOF. Let $\mu \neq 0$ be in $M(H)$, and let $\xi \in \hat{H}_G$. Write $\mu = \mu_G + \mu_I$ where $\mu_G \in M(H_G)$ and $\mu_I \in I$ using the previously described Raikov system. We will show $|\hat{\mu}_G(\xi)| \leq \|\hat{\mu}\|_\infty$.

We may assume $\text{spt } \mu_G$ (spt denotes the support) is compact in H_G . By Proposition 2, there is $p \in P_c(H_G)$ such that $p(0) = 1$ and $|p - 1| < \varepsilon/\|\mu\|$ on $\text{spt } \mu_G$.

Since $|\mu_I|(\text{spt } p) = 0$, we may assume $\text{spt } \mu_I \cap \text{spt } p = \emptyset$. Since p is uniformly continuous in the H_G -topology, there is a H_G -open neighborhood of 0, U , such that for $x \in H_G$ and $y \in U$, $|p(x+y) - p(x)| < \varepsilon/\|\mu\|$. Let $K = -K$ be a compact subset of H_G containing $\text{spt } p$ and $\text{spt } \mu_G$. By Proposition 3, choose V to be an H -open neighborhood of 0 such that $V = -V$ and $(x+V) \cap K \subset x+U$ for all $x \in K$; we further assume that $(\text{spt } p + V) \cap (\text{spt } \mu_I + V) = \emptyset$.

Now choose $\gamma \in \hat{H}$ by Proposition 4 such that $|\gamma - \bar{\xi}| < \varepsilon/\|\mu\|$ on K ; and choose $g \in P_c(H)$ with $\text{spt } g \subset V$, $g \geq 0$, and $\int_U g d\lambda = 1$. For any $x \in K$, $|(g * p d\lambda)(x) - p(x)| = |\int_V g(y)p(x-y) d\lambda(y) - p(x)| = |\int_U g(y)(p(x-y) - p(x)) d\lambda(y)| < \varepsilon/\|\mu\|$ (since $V \cap (x-K) \subset U$, $x \in \text{spt } p$). Thus letting $f = g * p d\lambda$, $\text{spt } f \subset V + \text{spt } p$ and $f \in P_c(H)$ (by Proposition 1). Also $f(0) < p(0) + \varepsilon/\|\mu\| = 1 + \varepsilon/\|\mu\|$, and $\text{spt } f \cap \text{spt } \mu_I = \emptyset$. For $x \in \text{spt } \mu_G$,

$$|f(x) - 1| \leq |f(x) - p(x)| + |p(x) - 1| < 2\varepsilon/\|\mu\|.$$

And

$$\begin{aligned} \left| \int_{H_G} \bar{\xi} d\mu_G - \int_H \gamma f d\mu \right| &\leq \left| \int_{H_G} \bar{\xi} d\mu_G - \int_{H_G} \gamma d\mu_G \right| \\ &\quad + \left| \int_{H_G} \gamma d\mu_G - \int_H \gamma f d\mu_G \right| + \left| \int_H \gamma f d\mu_I \right| \\ &< (\varepsilon/\|\mu\|) \|\mu_G\| + (2\varepsilon/\|\mu\|) \|\mu_G\| + 0 \\ &\leq 3\varepsilon. \end{aligned}$$

Now $|\int_H \gamma f d\mu| \leq f(0) \|\hat{\mu}\|_\infty < (1 + \varepsilon/\|\mu\|) \|\hat{\mu}\|_\infty$ (since γf is positive definite).

Summarizing, given $\xi \in \hat{H}_G$,

$$\begin{aligned} |\hat{\mu}_G(\xi)| &= \left| \int_{H_G} \bar{\xi} d\mu_G \right| \leq \left| \int_H \gamma f d\mu \right| + 3\varepsilon \\ &\leq (1 + \varepsilon/\|\mu\|) \|\hat{\mu}\|_\infty + 3\varepsilon \leq \|\hat{\mu}\|_\infty + 4\varepsilon. \end{aligned}$$

And so $\|\hat{\mu}_G\|_\infty \leq \|\hat{\mu}\|_\infty$. \square

COROLLARY 6. Let $\mathcal{M}(\hat{H})$, $\mathcal{M}(\hat{H}_G)$ and \mathfrak{J} denote the uniform closures of the Fourier-Stieltjes transforms of $M(H)$, $M(H_G)$, and I respectively. Then $\mathcal{M}(\hat{H}) = \mathcal{M}(\hat{H}_G) \oplus \mathfrak{J}$.

COROLLARY 7. If $\mu \in M(H)$ and $\hat{\mu} \in \mathcal{M}(\hat{H}_G)$, then $\mu \in M(H_G)$.

COROLLARY 8. Let \hat{H}_G be embedded in $\kappa\hat{H}$ (the maximal ideal space of $\mathcal{M}(\hat{H})$); equivalently, the closure of \hat{H} in Δ , by $\gamma \mapsto \pi_\gamma$ from \hat{H}_G to $\kappa\hat{H}$ where $\pi_\gamma(\mu) = \hat{\mu}_G(\gamma)$ ($\mu \in M(H)$). Since $\pi_\gamma(\mu) = 0$ for $\mu \in L^1(H)$ (recall G is nonopen in H), $\pi_\gamma \in \kappa\hat{H} \setminus \hat{H}$ (the fringe of \hat{H}). In particular, for $\mu \in M(H)$, $\|\hat{\mu}_G\|_\infty = \limsup \|\hat{\mu}_G\|_\infty \cong \|\hat{\mu}\|_\infty$.

These corollaries follow from the inequality $\|\hat{\mu}_G\|_\infty \leq \|\hat{\mu}\|$ ($\mu \in M(H)$). The proofs are discussed in a more general setting in [3]. For $G = \{0\}$ (and thus $H_G = H_d$), Corollary 7 is due to E. Hewitt for H with a restricted hypothesis and to W. Eberlein in general. A reference for these facts, plus a different (although closely related) direct sum decomposition, is [1].

Some interesting examples of LCA groups H with a nonopen subgroup G are: (1) H nondiscrete and $G = \{0\}$, (2) G noncompact and $H = \beta G$ the Bohr compactification of G , (3) $G = \mathbb{R}$ (the real numbers) and H a compact solenoidal group, and (4) certain local direct product groups embedded in the appropriate complete direct product groups.

BIBLIOGRAPHY

1. R. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970.
2. C. Dunkl and D. Ramirez, *C*-algebras generated by measures*, Bull. Amer. Math. Soc. **77** (1971), 411-412.
3. ———, *C*-algebras generated by Fourier-Stieltjes transforms*, Trans. Amer. Math. Soc. (to appear).
4. ———, *Homomorphisms on groups and induced maps on certain algebras of measures*, Trans. Amer. Math. Soc. **160** (1971), 475-485.
5. E. Hewitt, *The asymmetry of certain algebras of Fourier-Stieltjes transforms*, Michigan Math. J. **5** (1958), 149-158. MR **21** #4993.
6. H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford Math. Monographs, Clarendon Press, Oxford, 1968.
7. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR **27** #2808.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903