

WEAKLY ALMOST PERIODIC FUNCTIONALS CARRIED BY HYPERCOSETS⁽¹⁾

BY

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Abstract. For G a compact group and H a closed normal subgroup, we show that a weakly almost periodic (w.a.p.) linear functional on the Fourier algebra of G/H lifts to a w.a.p. linear functional on the Fourier algebra of G .

In the case of a compact abelian group G , the dual of a closed subgroup can be identified with a quotient group of the whole dual \hat{G} . If G is not abelian, and H is a closed normal subgroup, then an identification space, \tilde{H} , of the dual of H , \hat{H} , can be identified with a hypercoset structure on \hat{G} . Let H^\perp be the set of elements of \hat{G} whose kernel contains H . (Recall \hat{G} is the set of equivalence classes of continuous unitary irreducible representations of G .) Then \tilde{H} is identified with the set of hypercosets of H^\perp , with the trivial representation $H \rightarrow \{1\}$ of course identified with H^\perp itself. As in the abelian case, the Fourier algebra $A(G)$ of G splits into a direct sum of $A(G/H)$ -modules, one for each hypercoset of H^\perp . Again $A(G/H)$ itself corresponds to H^\perp . We show here that each of these modules is finitely generated, and use this result to show that weakly almost periodic (w.a.p.) linear functionals on $A(G/H)$ lift to w.a.p. linear functionals on $A(G)$ (the set of such is denoted $W(\hat{G})$).

We show that if G has an infinite abelian homomorphic image, then the space of Fourier-Stieltjes transforms of measures on G is not dense in $W(\hat{G})$, and $W(\hat{G})$ is not equal to $\mathcal{L}^\infty(\hat{G})$, the dual of $A(G)$. We will use some of the methods developed in our previous paper on w.a.p. functionals [6].

1. Notation and hypercosets. Let G be a compact nonabelian group. Using our previous notation [3, Chapters 7, 8] we let \hat{G} denote the set of equivalence classes of continuous unitary irreducible representations of G . For $\alpha \in \hat{G}$, choose $T_\alpha \in \alpha$, then T_α is a continuous homomorphism of G into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices where n_α is the dimension of α . We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $1 \leq i, j \leq n_\alpha$, and $T_{\alpha ij}$ to denote the (continuous) function $x \mapsto T_\alpha(x)_{ij}$. Let $V_\alpha = \text{Sp} \{T_{\alpha ij} : 1 \leq i, j \leq n_\alpha\}$ (where Sp denotes the linear span), then V_α is an n_α^2 -dimensional space of continuous functions invariant under left

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and right translation by G . Further let $\chi_\alpha(x) = \text{trace}(T_\alpha(x)) = \sum_{i=1}^n T_\alpha(x)_{ii}$. The function χ_α is called the character of α and it, as well as V_α , is independent of the choice of $T_\alpha \in \alpha$.

For $\alpha, \beta \in \hat{G}$ one can form the tensor product $T_\alpha \otimes T_\beta$ of the two representations. This tensor product decomposes into irreducible components: $T_\alpha \otimes T_\beta \cong \sum_{\gamma \in \hat{G}} M_{\alpha\beta}(\gamma) T_\gamma$, where $M_{\alpha\beta}(\gamma) = \int_G \chi_\alpha \chi_\beta \bar{\chi}_\gamma dm_G$, a nonnegative integer (m_G is the normalized Haar measure on G). This decomposition can also be written in the form $\chi_\alpha \chi_\beta = \sum_\gamma M_{\alpha\beta}(\gamma) \chi_\gamma$ (a finite sum). For $E, F \subset \hat{G}$, we define

$$E \otimes F = \{\gamma \in \hat{G} : M_{\alpha\beta}(\gamma) \neq 0, \text{ some } \alpha \in E, \beta \in F\}.$$

This operation makes \hat{G} into a hypergroup. For each $\alpha \in \hat{G}$, there is a conjugate $\bar{\alpha} \in \hat{G}$ such that $\chi_{\bar{\alpha}}(x) = (\chi_\alpha(x))^-$ ($x \in G$). If $E \subset \hat{G}$ and $E \otimes E \subset E$, then E is called a subhypergroup, and if further $\bar{E} = \{\bar{\alpha} : \alpha \in E\} \subset E$ then E is called a normal subhypergroup.

For any set $S \subset G$, let $S^\perp = \{\alpha \in \hat{G} : S \subset \text{kernel } T_\alpha\}$ then S^\perp is a normal subhypergroup. For $E \subset \hat{G}$, let $E^\perp = \bigcap_{\alpha \in E} (\text{kernel } T_\alpha)$, a closed normal subgroup of G . Helgason [7] has shown that if H is a closed normal subgroup of G then $(H^\perp)^\perp = H$.

If E is a normal subhypergroup of \hat{G} and $\alpha \in \hat{G}$ then $\alpha \otimes E$ is called a hypercoset of E . We will prove later that \hat{G} is the disjoint union of hypercosets of E .

Let X be an n -dimensional complex inner product space. Let $\mathcal{B}(X)$ be the space of linear maps: $X \rightarrow X$. The operator norm of $A \in \mathcal{B}(X)$ is defined to be $\|A\|_\infty = \sup \{ \|A\xi\| : \xi \in X, \|\xi\| \leq 1\}$. The trace of A is defined to be $\text{Tr } A = \sum_{i=1}^n (A\xi_i, \xi_i)$ where $\{\xi_i\}_{i=1}^n$ is any orthonormal basis for X and (\cdot, \cdot) is the inner product in X . We define the dual norm to $\|\cdot\|_\infty$ by

$$\|A\|_1 = \sup \{ |\text{Tr}(AB)| : B \in \mathcal{B}(X), \|B\|_\infty \leq 1 \}.$$

One can show that $\|A\|_1 = \text{Tr}(|A|)$, where $|A| = (A^*A)^{1/2}$.

Let ϕ be a set $\{\phi_\alpha : \alpha \in \hat{G}, \phi_\alpha \in \mathcal{B}(C^{n_\alpha}), \sup_\alpha \|\phi_\alpha\|_\infty < \infty\}$. The set of all such ϕ is denoted by $\mathcal{L}^\infty(\hat{G})$. It is a C^* -algebra under the norm $\|\phi\|_\infty = \sup \{\|\phi_\alpha\|_\infty : \alpha \in \hat{G}\}$ and coordinatewise operations (* denotes the operator adjoint).

Let $\mathcal{L}^1(\hat{G}) = \{\phi \in \mathcal{L}^\infty(\hat{G}) : \|\phi\|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \|\phi_\alpha\|_1 < \infty\}$. Then $\mathcal{L}^1(\hat{G})$ with the norm $\|\cdot\|_1$ is a Banach space and its dual may be identified with $\mathcal{L}^\infty(\hat{G})$ under the pairing $\langle \phi, \psi \rangle = \sum_\alpha n_\alpha \text{Tr}(\phi_\alpha \psi_\alpha)$ ($\phi \in \mathcal{L}^1(\hat{G}), \psi \in \mathcal{L}^\infty(\hat{G})$). Let $\mu \in M(G)$, the measure algebra of G , then the Fourier transform of μ , $\hat{\mu}$, is the function $\alpha \mapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1}) d\mu(x)$ ($\alpha \in \hat{G}$), and $\hat{\mu} \in \mathcal{L}^\infty(\hat{G})$ with $\|\hat{\mu}\|_\infty \leq \|\mu\|$. If $f \in C(G)$ (the continuous functions on G), then $\hat{f}_\alpha = \int_G T_\alpha(x^{-1}) f(x) dm_G(x)$ ($\alpha \in \hat{G}$).

We will now define $A(G)$, the Fourier algebra of G . Let

$$A(G) = \{f \in C(G) : \hat{f} \in \mathcal{L}^1(\hat{G})\},$$

then $A(G)$ is in fact isomorphic to $\mathcal{L}^1(\hat{G})$, since for $\phi \in \mathcal{L}^1(\hat{G})$ the function $f(x) = \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr}(\phi_\alpha T_\alpha(x))$ ($x \in G$) is continuous and $\hat{f} = \phi$. Put $\|f\|_A = \|\hat{f}\|_1$. Further

$A(G)$ is a (commutative) Banach algebra under the pointwise operations on G ; and its dual is $\mathcal{L}^\infty(\hat{G})$, under the pairing

$$\langle f, \phi \rangle = \sum_{\alpha} n_{\alpha} \text{Tr}(f_{\alpha} \phi_{\alpha}) \quad (f \in A(G), \phi \in \mathcal{L}^\infty(\hat{G}));$$

for proofs see [3, p. 93].

DEFINITION 1.1. For $\phi \in \mathcal{L}^\infty(\hat{G})$ define the carrier of ϕ , $\text{cr } \phi = \{\alpha \in \hat{G} : \phi_{\alpha} \neq 0\}$. For $E \subset \hat{G}$, let $\mathcal{L}^\infty(E) = \{\phi \in \mathcal{L}^\infty(\hat{G}) : \text{cr } \phi \subset E\}$, and let $A(E) = \{f \in A(G) : \text{cr } \hat{f} \subset E\}$. In fact $A(E)$ is the closed span of $\{V_{\alpha} : \alpha \in E\}$.

PROPOSITION 1.2. *The spaces $A(E)$, $E \subset \hat{G}$, are exactly the closed subspaces of $A(G)$ which are invariant under left and right translation by G . The dual of $A(E)$ is $\mathcal{L}^\infty(E)$.*

PROPOSITION 1.3. *Let $E, F \subset \hat{G}$, then the closed linear span of*

$$\{fg : f \in A(E), g \in A(F)\}$$

is equal to $A(E \otimes F)$.

COROLLARY 1.4. *For $E \subset \hat{G}$, $A(E)$ is a subalgebra of $A(G)$ if and only if E is a subhypergroup. Further $A(E)$ is a conjugate-closed ($f \mapsto \bar{f}$) subalgebra of $A(G)$ if and only if E is a normal subhypergroup, and in that case*

$$A(E) = \{f \in A(G) : f(h_1 x h_2) = f(x), \text{ for all } h_1, h_2 \in E^\perp, x \in G\},$$

the functions in $A(G)$ constant on cosets of E^\perp , a closed normal subgroup of G .

COROLLARY 1.5. *If E is a finite subhypergroup of \hat{G} then E is normal.*

Proof. In fact $A(E)$ is a finite dimensional subalgebra of a conjugate closed algebra $A(G)$ and is thus itself conjugate-closed (since the maximal ideal space of $A(E)$ is a finite set). \square

REMARK 1.6. The Fourier algebra of a compact group G is the subject of [3, Chapter 8]. Helgason [7] constructed the duality between normal subhypergroups of \hat{G} and closed normal subgroups of G . Translation-invariant uniformly closed linear subspaces of $C(G)$ are discussed by Rider in [9]. What is observed by Iltis [8], that finite subhypergroups are normal, is implicit in Rider [9, p. 980].

2. Restrictions of representations to normal subgroups. In this section, G denotes a compact group, and H denotes a closed normal subgroup of G . Define \hat{H} similarly to \hat{G} , and denote the character of $\gamma \in \hat{H}$ by ξ_{γ} , and let $T_{\gamma}^H \in \gamma$. Denote the normalized Haar measure of H by m_H . There exists a natural homomorphism of G into the group of automorphisms of H ; namely, for $x \in G$, let $S_x h = x h x^{-1}$ ($h \in H$), then S_x is an automorphism of H . Each S_x induces a permutation \hat{x} on \hat{H} such that $\xi_{\hat{x}\gamma}(h) = \xi_{\gamma}(S_x h)$ ($h \in H$). Now define an equivalence relation on \hat{H} by $\gamma_1 \sim \gamma_2$ if and only if $\gamma_2 = \hat{x}\gamma_1$ for some $x \in G$ ($\gamma_1, \gamma_2 \in \hat{H}$). Denote the set of such equivalence classes by \tilde{H} . Let $\alpha \in \tilde{H}$ then $T_{\alpha}|_H$ is a continuous unitary representation of H and thus decomposes:

$$T_{\alpha}|_H = \sum_{\gamma \in \hat{H}} a_{\gamma} T_{\gamma}^H$$

where the a_γ 's are nonnegative integers and only finitely many are nonzero.

THEOREM 2.1. *Let $\alpha \in \hat{G}$, then there is a positive integer N_α and a class $\Gamma_\alpha \in \hat{H}$ such that*

$$\chi_\alpha|H = N_\alpha \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma.$$

Further each equivalence class in \hat{H} is finite.

Proof. This theorem is nothing but the compact group analogue to the well-known finite groups result (see e.g. [1, p. 278]). We sketch an argument. Write $\chi_\alpha|H = \sum_\gamma c_\gamma \xi_\gamma$. Now $\chi_\alpha|H$ is invariant under each S_x , $x \in G$, thus if $\gamma_1 \sim \gamma_2$ then $c_{\gamma_1} = c_{\gamma_2}$. It remains to show that if $c_{\gamma_1} \neq 0$ and $c_{\gamma_2} \neq 0$ then $\gamma_1 \sim \gamma_2$. Now $d\mu = \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma dm_H$ is a central measure in $M(G)$, and for any $\alpha \in \hat{G}$, μ is orthogonal to either all or none of the diagonal entry functions. If $\gamma' \notin \Gamma_\alpha$, then

$$\int_H \xi_{\gamma'} \left(\sum_{\gamma \in \Gamma_\alpha} \xi_\gamma \right) dm_H = 0.$$

Thus the class $\Gamma_\alpha \in \hat{H}$ is uniquely determined and is evidently finite. However any $\gamma \in \hat{H}$ appears in the restriction of some $\alpha \in \hat{G}$ (induced representation argument) and thus any class in \hat{H} is finite. \square

COROLLARY 2.2. *For any $\alpha, \beta \in \hat{G}$, either $n_\beta \chi_\alpha|H = n_\alpha \chi_\beta|H$ or $\int_H \chi_\alpha \bar{\chi}_\beta dm_H = 0$.*

Proof. For $\alpha, \beta \in \hat{G}$, if $\Gamma_\alpha = \Gamma_\beta$ then $n_\alpha/N_\alpha = \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma(e) = n_\beta/N_\beta$ (e is identity in G). If $\Gamma_\alpha \neq \Gamma_\beta$, then $\int_H \chi_\alpha \bar{\chi}_\beta dm_H = 0$ by the orthogonality relations for characters of H . \square

REMARK 2.3. Rider uses this corollary in [10].

REMARK 2.4. For $\alpha \in \hat{G}$, $\alpha \in H^\perp$ if and only if $\Gamma_\alpha = \{\{1\}\}$ ($\{1\}$ denotes the trivial representation $H \rightarrow \{1\}$), and in this case, $\chi_\alpha|H = n_\alpha = N_\alpha$.

THEOREM 2.5. *For $\alpha, \beta \in \hat{G}$, $\alpha \in \beta \otimes H^\perp$ if and only if $\Gamma_\alpha = \Gamma_\beta$. Thus \hat{G} is split into disjoint hypercosets, and these hypercosets are indexed by \hat{H} .*

Proof. Let $\alpha \in \beta \otimes H^\perp$, then there exists some $\delta \in H^\perp$ such that $M_{\beta\delta}(\alpha) \neq 0$, that is $\chi_\beta \chi_\delta = \chi_\alpha + \phi$, where ϕ is some nonnegative integer combination of characters. Now restrict to H to obtain

$$n_\delta N_\beta \sum_{\gamma \in \Gamma_\beta} \xi_\gamma = N_\alpha \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma + \phi|H,$$

thus $\Gamma_\alpha \subset \Gamma_\beta$, hence $\Gamma_\alpha = \Gamma_\beta$.

Conversely if $\Gamma_\alpha = \Gamma_\beta$, then $(\chi_\alpha|H)(\chi_\beta|H)^{-1} = 1 + \phi$ (some ϕ as above). (This follows from the relation $M_{\gamma\gamma}(\{1\}) = \int_H |\xi_\gamma|^2 dm_H = 1$ for any $\gamma \in \hat{H}$.) This implies that $(\alpha \otimes \bar{\beta}) \cap H^\perp \neq \emptyset$. Thus there is a $\delta \in H^\perp$ such that $M_{\alpha\bar{\beta}}(\delta) \neq 0$, but

$$M_{\alpha\bar{\beta}}(\delta) = \int_G \chi_\alpha \bar{\chi}_\beta \bar{\chi}_\delta dm_G = \int_G \bar{\chi}_\alpha \chi_\beta \chi_\delta dm_G = M_{\beta\delta}(\alpha)$$

and so $\alpha \in \beta \otimes H^\perp$. \square

Observe that the Fourier algebra of G/H is isomorphic to a closed subalgebra of $A(G)$, namely $A(H^\perp)$ (which we will denote by A_H), since $(G/H)^\wedge$ may be identified with H^\perp . We will now decompose $A(G)$ into a direct sum of A_H -modules.

THEOREM 2.6. *To each $\Gamma \in \tilde{H}$ there corresponds a closed subspace A_Γ of $A(G)$ which is also an A_H -module. Further each $f \in A(G)$ has a unique decomposition $f = \sum_{\Gamma \in \tilde{H}} f_\Gamma$, where $f_\Gamma \in A_\Gamma$ and $\|f\|_A = \sum_{\Gamma \in \tilde{H}} \|f_\Gamma\|_A$. Also the A_Γ 's are the minimal closed left and right translation invariant A_H -submodules of $A(G)$.*

Proof. For $\Gamma \in \tilde{H}$, let $E_\Gamma = \{\alpha \in \hat{G} : \Gamma_\alpha = \Gamma\}$, that is, the hypercoset of H corresponding to Γ . Then put $A_\Gamma = A(E_\Gamma)$. Clearly \hat{G} is the disjoint union of $\{E_\Gamma\}_{\Gamma \in \tilde{H}}$, so the decomposition of $A(G)$ follows from the obvious decomposition of $\mathcal{L}^1(\hat{G})$. Let $\Gamma \in \tilde{H}$ and choose $\alpha \in E_\Gamma$ then $E_\Gamma = \alpha \otimes H^\perp$, so that $E_\Gamma \otimes H^\perp = E_\Gamma$ and thus, by Proposition 1.3, $A_H \cdot A_\Gamma \subset A_\Gamma$. So A_Γ is a closed A_H -submodule of $A(G)$. If a nontrivial closed left and right translation invariant A_H -module is contained in A_Γ , then it is determined by some nonempty subset $F \subset E_\Gamma$. But if $\alpha \in F$ and $\beta \in H^\perp$ then $\alpha \otimes \beta \in F$, thus F is a hypercoset, hence equals E_Γ . \square

REMARK 2.7. If G is abelian then each A_Γ has a single generator (in the algebraic as well as the topological sense) over A_H . In the general case for $\Gamma \in \tilde{H}$ and some $\alpha \in E_\Gamma$, the functions $\{T_{\alpha_{ij}} : 1 \leq i, j \leq n_\alpha\}$ generate A_Γ topologically, but it is not clear that they do so algebraically. However the following is true.

THEOREM 2.8. *Let $\Gamma \in \tilde{H}$, then A_Γ is a finitely generated A_H -module, that is there exists $g_1, \dots, g_m \in A_\Gamma$ (some $m < \infty$) so that each $f \in A_\Gamma$ may be written as $f = \sum_{i=1}^m k_i g_i$, with $k_i \in A_H$. Further there exists a constant $M < \infty$, such that the functions k_i may be chosen with $\|k_i\|_A \leq M \|f\|_A$.*

Proof. In a paper of Dunkl [2] the following is shown: let τ be a continuous unitary representation of H on a finite dimensional space V , and let $A(G, V)$ be the space of V -valued functions on G with each coordinate function in $A(G)$. Define

$$M(\tau) = \{f \in A(G, V); f(hx) = \tau(h)f(x) \text{ for all } h \in H, x \in G\},$$

denoted $A(\tau)$ in [2]. Then $M(\tau)$ is a finitely generated (algebraically) A_H -module.

We now point out the applicability of this theorem to the present situation. Pick $\alpha \in \Gamma$, and let $V = V_\alpha|H$. Recall $V_\alpha = \text{Sp} \{T_{\alpha_{ij}} : 1 \leq i, j \leq n_\alpha\}$ so that V is a finite dimensional space of continuous functions on H , and is in fact the left and right translation invariant (by H) space generated by $\{\xi_\gamma : \gamma \in \Gamma\}$. This shows that V depends only on Γ , that for any $f \in A_\Gamma$, $f|H \in V$, and finally that V is invariant under each S_x , $x \in G$ (that is, if $g \in V$, $x \in G$, then the function $h \mapsto g(xhx^{-1})$ is in V ($h \in H$)). Observe that a continuous unitary representation τ of H is realized on V , namely right translation, with the inner product on V given by $(f, g)_H = \int_H f \bar{g} \, dm_H$ ($f, g \in V$), and $\tau(h)f(h_1) = f(h_1h)$ ($f \in V, h, h_1 \in H$).

We claim that $M(\tau) = A_\Gamma$, in fact if $f \in A_\Gamma$ then assign to each $x \in G$ the function $f(x, \cdot) : h_1 \mapsto f(h_1x) = (R(x)f)(h_1)$. Now A_Γ is invariant under the right translation

$R(x)$ so $R(x)f|H \in V$, thus $f(x, \cdot) \in V$. Further for $x \in G$, $h \in H$, $f(hx, h_1) = f(h_1, hx) = f(x, h_1h) = \tau(h)f(x, h_1)$ ($h_1 \in H$), that is, $f(hx, \cdot) = \tau(h)f(x, \cdot)$. Finally to check the coordinate functions of $f(x, \cdot)$ let $g \in V$ and consider the function $x \mapsto (f(x, \cdot), g)_H = \int_H f(hx)(g(h))^{-1} dm_H = \mu * f(x)$, where μ is the measure $(g(h^{-1}))^{-1} dm_H(h)$, and so $\mu * f \in A(G)$. Conversely, if $f \in M(\tau)$, so f is of the form $f(x, h)$, with $f(x, \cdot) \in V$, then put $f(x) = f(x, e)$. Thus $f \in A(G)$ (by finite dimensionality of V , point evaluation is a bounded linear functional). Further for each $x \in G$ let $g = R(x)f|H$, then $g(h) = f(hx) = f(hx, e) = \tau(h)f(x, e) = f(x, h)$ so the function $g \in V$, thus $f \in A_\Gamma$. Hence $A_\Gamma = M(\tau)$ and thus there exist generators $g_1, \dots, g_m \in A_\Gamma$ (some $m < \infty$).

Now consider the bounded linear map $T: A_H \times A_H \times \dots \times A_H$ (m copies) $\rightarrow A_\Gamma$ defined by $T(k_1, \dots, k_m) = \sum_{i=1}^m k_i g_i$. By the above paragraph T is onto and so by the open mapping theorem there exists $M < \infty$ such that $\{T(k_1, \dots, k_m) : \|k_j\|_A \leq M\} \supset \{f \in A_\Gamma : \|f\|_A \leq 1\}$. \square

3. Homomorphisms. Let π be a continuous homomorphism of a compact group G into a compact group K , and let H be the kernel of π . Then π induces the map $\pi_1: C(K) \rightarrow C(G)$, given by $\pi_1 f(x) = f(\pi x)$, $f \in C(K)$, $x \in G$. The adjoint of π_1 , denoted by π^* , takes $M(G)$ into $M(K)$. Further π_1 maps $A(K)$ into $A(G)$, since $A(K)$ is spanned by the continuous positive definite functions and these are preserved by π_1 . Also $\pi_1|A(K)$ is a bounded operator on $A(K)$ since each $f \in A(K)$ is a sum $f = f_1 - f_2 + i(f_3 - f_4)$, f_i positive definite and $\sum_{i=1}^4 f_i(e) \leq 2\|f\|_A$. Finally the adjoint of $\pi_1|A(K)$ is a bounded map $\hat{\pi}: \mathcal{L}^\infty(\hat{G})$ into $\mathcal{L}^\infty(\hat{K})$. Let $\mathcal{M}(\hat{G})$, $\mathcal{M}(\hat{K})$ be the closures of $M(G)^\wedge$, $M(K)^\wedge$ in $\mathcal{L}^\infty(\hat{G})$ and $\mathcal{L}^\infty(\hat{K})$ respectively.

PROPOSITION 3.1. $\hat{\pi}\mathcal{M}(\hat{G}) \subset \mathcal{M}(\hat{K})$.

Proof. Let $\mu \in M(G)$, then $\hat{\mu}$ satisfies the following: $\langle f, \hat{\mu} \rangle = \int_G f(x^{-1}) d\mu(x)$, $f \in A(G)$. Now let $g \in A(K)$, then

$$\langle g, \hat{\pi}\hat{\mu} \rangle = \langle \pi_1 g, \hat{\mu} \rangle = \int_G g(\pi x^{-1}) d\mu(x) = \int_K g(k^{-1}) d\pi^*\mu(k) = \langle g, (\pi^*\mu) \rangle.$$

Thus $\hat{\pi}\hat{\mu} = (\pi^*\mu)^\wedge \in M(K)^\wedge$. The continuity of $\hat{\pi}$ finishes the proof. \square

Observe that π factors into $G \rightarrow G/H \rightarrow K$, where G/H is identified with a closed subgroup of K . Further $M(G/H)$ is identified with a closed subalgebra of $M(G)$, namely $m_H * M(G)$ (note m_H is an idempotent, see [3, Chapter 9]). Also $\mathcal{L}^\infty((G/H)^\wedge) \cong \mathcal{L}^\infty(H^\perp)$ and $\mathcal{M}((G/H)^\wedge) \cong \mathcal{M}(\hat{G}) \cap \mathcal{L}^\infty(H^\perp)$ (since \hat{m}_H is the projection of $\mathcal{L}^\infty(\hat{G})$ onto $\mathcal{L}^\infty(H^\perp)$).

Finally $\hat{\pi}$ takes $\mathcal{M}(\hat{G})$ onto $\mathcal{M}(\hat{K})$, or $\mathcal{L}^\infty(\hat{G})$ onto $\mathcal{L}^\infty(\hat{K})$ if and only if π maps G onto K , for otherwise πG is a proper closed subgroup of K , and $\phi \in \hat{\pi}\mathcal{L}^\infty(\hat{G})$ if and only if $\text{spt } \phi \subset \pi G$ (where the support of ϕ , $\text{spt } \phi$, is the least compact subset $E \subset K$ with the property that $f \in A(K)$, $f = 0$ on a neighborhood of E implies $\langle f, \phi \rangle = 0$).

Now we investigate the effect of $\hat{\pi}$ on $W(\hat{G})$, the weakly almost periodic (w.a.p.) elements of $\mathcal{L}^\infty(\hat{G})$. We state some appropriate definitions and results from our previous paper [6].

PROPOSITION 3.2. $\mathcal{L}^\infty(\hat{G})$ is an $A(G)$ -module. The action is defined by $\langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle$ ($f, g \in A(G), \phi \in \mathcal{L}^\infty(\hat{G})$), and $\|f \cdot \phi\|_\infty \leq \|f\|_A \|\phi\|_\infty$. Further $\text{cr}(f \cdot \phi) \subset (\text{cr } f)^- \otimes \text{cr } \phi$.

DEFINITION 3.3. For $\phi \in \mathcal{L}^\infty(\hat{G})$, one says that ϕ is weakly almost periodic if the map $f \mapsto f \cdot \phi$ is a weakly compact operator of $A(G)$ into $\mathcal{L}^\infty(\hat{G})$ ($f \in A(G)$). The set of all such ϕ is denoted by $W(\hat{G})$.

DEFINITION 3.4. Let $B = \{f \in A(G) : \|f\|_A \leq 1\}$. For $\alpha \in \hat{G}$, let $B_\alpha = B \cap V_\alpha$. Let $E = \bigcup_{\alpha \in \hat{G}} B_\alpha$.

Some properties of $W(\hat{G})$ (see [6]):

- (1) $W(\hat{G})$ is a closed submodule of $\mathcal{L}^\infty(\hat{G})$.
- (2) For $\phi \in \mathcal{L}^\infty(\hat{G})$ to be in $W(\hat{G})$ it is necessary and sufficient that $\{f_n \cdot \phi\}$ have a weakly convergent subsequence for any sequence $\{f_n\} \subset E$ (also true if E is replaced by B).

THEOREM 3.5. Let π be a continuous homomorphism of G into K (compact groups). Then $\hat{\pi}W(\hat{G}) \subset W(\hat{K})$.

Proof. Suppose $\phi \in W(\hat{G})$, and $\{f_n\}$ is a bounded sequence in $A(K)$. Then $\{\pi_1 f_n\}$ is a bounded sequence in $A(G)$, and there exists a subsequence such that $(\pi_1 f_{n_j}) \cdot \phi$ converges weakly to $\psi \in \mathcal{L}^\infty(\hat{G})$. But $\hat{\pi}((\pi_1 f_{n_j}) \cdot \phi) = f_{n_j} \cdot (\hat{\pi}\phi)$, so $f_{n_j} \cdot \hat{\pi}\phi$ converges weakly to $\hat{\pi}\psi \in \mathcal{L}^\infty(\hat{K})$ (for $\hat{\pi}$, being strongly continuous, is weakly continuous). Hence $\hat{\pi}\phi \in W(\hat{K})$. \square

Henceforth we assume π is onto K so we identify \hat{K} with H^\perp , and $\mathcal{L}^\infty(\hat{K})$ with $\mathcal{L}^\infty(H^\perp)$. We have just seen that the restriction map $\hat{\pi}: \mathcal{L}^\infty(\hat{G}) \rightarrow \mathcal{L}^\infty(H^\perp)$ takes $W(\hat{G})$ into $W(\hat{K})$. We will now show that in fact $\hat{\pi}(W(\hat{G}) \cap \mathcal{L}^\infty(H^\perp)) = W(\hat{K})$.

DEFINITION 3.6. Let $\{\phi_n\}$ be a sequence in $\mathcal{L}^\infty(\hat{G})$. Say $\phi_n \xrightarrow{n} \phi \in \mathcal{L}^\infty(\hat{G})$ quasi-uniformly if $(\phi_n)_\alpha \xrightarrow{n} \phi_\alpha$ for each $\alpha \in \hat{G}$, and for each $\varepsilon > 0, N = 1, 2, 3, \dots$, there exist integers $m_1, \dots, m_k \geq N$, such that $\min_{1 \leq i \leq k} \|(\phi_{m_i})_\alpha - \phi_\alpha\|_\infty < \varepsilon$ for each $\alpha \in \hat{G}$.

THEOREM 3.7 [6]. Let $\{\phi_n\} \subset \mathcal{L}^\infty(\hat{G})$. Then $\phi_n \xrightarrow{n} \phi \in \mathcal{L}^\infty(\hat{G})$ weakly if and only if $\sup_n \|\phi_n\|_\infty < \infty$, and every subsequence of $\{\phi_n\}$ converges quasi-uniformly to ϕ .

THEOREM 3.8. Let $\phi \in W(\hat{K})$, that is, $\phi \in \mathcal{L}^\infty(H^\perp)$, and for each bounded sequence, $\{f_n\} \subset A(K) = A_H$ (see previous section), $\{f_n \cdot \phi\}$ has a weakly convergent subsequence. Then $\phi \in W(\hat{G})$ (note $\phi_\alpha = 0$ for $\alpha \notin H^\perp$).

Proof. Let $\{f_n\} \subset E = \bigcup_\alpha B_\alpha$, with $f_n \in B_{\alpha_n}, n = 1, 2, 3, \dots$. We must show that $\{f_n \cdot \phi\}$ has a weakly convergent subsequence. There are two possibilities for $\{\alpha_n\}$:

- (1) There are infinitely many distinct cosets $\bar{\alpha}_n \otimes H^\perp$. That is, there exists a subsequence f_{n_j} such that the sets $\text{cr}(f_{n_j} \cdot \phi) \subset \bar{\alpha}_{n_j} \otimes H^\perp$ are all disjoint. Then $f_{n_j} \cdot \phi \xrightarrow{j} 0$ weakly by Theorem 3.7.

(2) Infinitely many $\alpha_n \in \alpha \otimes H^\perp$, some $\alpha \in \hat{G}$. Thus there is a bounded subsequence f_{n_j} in A_Γ , where $\Gamma = \Gamma_\alpha$ (recall Theorem 2.6). By Theorem 2.8, there exist $g_1, \dots, g_m \in A_\Gamma$ and functions $h_{ij} \in A_H$, and $M < \infty$, such that $f_{n_j} = \sum_{i=1}^m h_{ij} g_i$, and $\|h_{ij}\|_A \leq M$, all i, j . By successively extracting subsequences from $\{h_{1j}\}, \{h_{2j}\}, \dots, \{h_{mj}\}$ and reindexing, we obtain $\psi_1, \dots, \psi_m \in \mathcal{L}^\infty(H^\perp)$ such that $h_{ij} \cdot \phi \xrightarrow{j} \psi_i$ weakly, $i=1, \dots, m$. The map $\psi \mapsto g_i \cdot \psi$ on \mathcal{L}^∞ is strongly, hence weakly continuous, thus

$$f_{n_j} \cdot \phi = \sum_{i=1}^m g_i \cdot (h_{ij} \cdot \phi) \xrightarrow{j} \sum_{i=1}^m g_i \cdot \psi_i \text{ weakly. } \square$$

COROLLARY 3.9. *If $W(\hat{K}) \neq \mathcal{L}^\infty(\hat{K})$ then $W(\hat{G}) \neq \mathcal{L}^\infty(\hat{G})$. If $\mathcal{M}(\hat{K}) \neq W(\hat{K})$ then $\mathcal{M}(\hat{G}) \neq W(\hat{G})$. (Recall from [6] that $\mathcal{M}(\hat{G}) \subset W(\hat{G})$.)*

COROLLARY 3.10. *If G has an infinite abelian image, then $\mathcal{M}(\hat{G}) \neq W(\hat{G}) \neq \mathcal{L}^\infty(\hat{G})$.*

Proof. If K is an infinite compact abelian group, then $\mathcal{M}(\hat{K}) \neq W(\hat{K}) \neq \mathcal{L}^\infty(\hat{K})$ (see [3, Chapter 4] and [6]). \square

REMARK 3.11. In [4] we show that $W(\hat{G}) \neq \mathcal{L}^\infty(\hat{G})$ for any infinite compact group G . In [5] we show that $\mathcal{M}(\hat{G}) \neq W(\hat{G})$ for any compact group G which contains an infinite abelian subgroup.

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