WEAKLY ALMOST PERIODIC FUNCTIONALS CARRIED BY HYPERCOSETS(1)

BY

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Abstract. For $G$ a compact group and $H$ a closed normal subgroup, we show that a weakly almost periodic (w.a.p.) linear functional on the Fourier algebra of $G/H$ lifts to a w.a.p. linear functional on the Fourier algebra of $G$.

In the case of a compact abelian group $G$, the dual of a closed subgroup can be identified with a quotient group of the whole dual $\hat{G}$. If $G$ is not abelian, and $H$ is a closed normal subgroup, then an identification space, $\hat{H}$, of the dual of $H$, $\hat{H}$, can be identified with a hypercoset structure on $\hat{G}$. Let $H^\perp$ be the set of elements of $\hat{G}$ whose kernel contains $H$. (Recall $\hat{G}$ is the set of equivalence classes of continuous unitary irreducible representations of $G$.) Then $\hat{H}$ is identified with the set of hypercosets of $H^\perp$, with the trivial representation $H \rightarrow \{1\}$ of course identified with $H^\perp$ itself. As in the abelian case, the Fourier algebra $A(G)$ of $G$ splits into a direct sum of $A(G/H)$-modules, one for each hypercoset of $H^\perp$. Again $A(G/H)$ itself corresponds to $H^\perp$. We show here that each of these modules is finitely generated, and use this result to show that weakly almost periodic (w.a.p.) linear functionals on $A(G/H)$ lift to w.a.p. linear functionals on $A(G)$ (the set of such is denoted $W(G)$).

We show that if $G$ has an infinite abelian homomorphic image, then the space of Fourier-Stieltjes transforms of measures on $G$ is not dense in $W(\hat{G})$, and $W(G)$ is not equal to $\mathcal{L}^\infty(\hat{G})$, the dual of $A(G)$. We will use some of the methods developed in our previous paper on w.a.p. functionals [6].

1. Notation and hypercosets. Let $G$ be a compact nonabelian group. Using our previous notation [3, Chapters 7, 8] we let $\hat{G}$ denote the set of equivalence classes of continuous unitary irreducible representations of $G$. For $\alpha \in \hat{G}$, choose $T_\alpha \in \alpha$, then $T_\alpha$ is a continuous homomorphism of $G$ into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices where $n_\alpha$ is the dimension of $\alpha$. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $1 \leq i, j \leq n_\alpha$, and $T_{\alpha ij}$ to denote the (continuous) function $x \mapsto T_\alpha(x)_{ij}$. Let $V_\alpha = \text{Sp} \{ T_{\alpha ij} : 1 \leq i, j \leq n_\alpha \}$ (where Sp denotes the linear span), then $V_\alpha$ is an $n_\alpha^2$-dimensional space of continuous functions invariant under left

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and right translation by $G$. Further let $\chi_\alpha(x) = \text{trace } (T_\alpha(x)) = \sum_{i=1}^n T_\alpha(x)_{ii}$. The function $\chi_\alpha$ is called the character of $\alpha$ and it, as well as $V_\alpha$, is independent of the choice of $T_\alpha \in \alpha$.

For $\alpha, \beta \in \hat{G}$ one can form the tensor product $T_\alpha \otimes T_\beta$ of the two representations. This tensor product decomposes into irreducible components: $T_\alpha \otimes T_\beta \cong \sum_{r \in \delta} M_{ab}(\gamma) T_\gamma$, where $M_{ab}(\gamma) = \int_\alpha \chi_\alpha(x) \chi_\beta(x) \, dm_\alpha$, a nonnegative integer ($m_\alpha$ is the normalized Haar measure on $G$). This decomposition can also be written in the form $\chi_\alpha \chi_\beta = \sum_{r} M_{ab}(\gamma) \chi_\gamma$ (a finite sum). For $E, F \subset \hat{G}$, we define

$$E \otimes F = \{ \gamma \in \hat{G} : M_{ab}(\gamma) \neq 0, \text{ some } \alpha \in E, \beta \in F \}. $$

This operation makes $\hat{G}$ into a hypergroup. For each $\alpha \in \hat{G}$, there is a conjugate $\tilde{\alpha} \in \hat{G}$ such that $\chi_{\tilde{\alpha}}(x) = (\chi_\alpha(x))^{-1} (x \in G)$. If $E \subset \hat{G}$ and $E \otimes E \subset E$, then $E$ is called a subhypergroup, and if further $\overline{E} = \{ \tilde{\alpha} : \alpha \in E \} \subset E$ then $E$ is called a normal subhypergroup.

For any set $S \subset G$, let $S^\perp = \{ \alpha \in \hat{G} : S \subset \text{kernel } T_\alpha \}$ then $S^\perp$ is a normal subhypergroup. For $E \subset \hat{G}$, let $E^\perp = \bigcap_{\alpha \in E} \text{(kernel } T_\alpha)$, a closed normal subgroup of $G$. Helgason [7] has shown that if $H$ is a closed normal subgroup of $G$ then $(H^\perp)^\perp = H$.

If $E$ is a normal subhypergroup of $\hat{G}$ and $\alpha \in \hat{G}$ then $\alpha \otimes E$ is called a hypercoset of $E$. We will prove later that $\hat{G}$ is the disjoint union of hypercosets of $E$.

Let $X$ be an $n$-dimensional complex inner product space. Let $\mathcal{B}(X)$ be the space of linear maps: $X \to X$. The operator norm of $A \in \mathcal{B}(X)$ is defined to be $\|A\|_\infty = \sup \{ |A\xi| : \xi \in X, |\xi| \leq 1 \}$. The trace of $A$ is defined to be $\text{Tr } A = \sum_{i=1}^n (A_{ii}, \xi_i)$ where $\{\xi_i\}_{i=1}^n$ is any orthonormal basis for $X$ and $(\cdot, \cdot)$ is the inner product in $X$. We define the dual norm to $\| \cdot \|_\infty$ by

$$\|A\|_1 = \sup \{ |\text{Tr } (AB)| : B \in \mathcal{B}(X), \|B\|_\infty \leq 1 \}. $$

One can show that $\|A\|_1 = \text{Tr } (|A|)$, where $|A| = (A^* A)^{1/2}$.

Let $\phi$ be a set $\{ \phi_\alpha : \alpha \in \hat{G}, \phi_\alpha \in \mathcal{B}(C^\infty_a), \sup_{\alpha} \|\phi_\alpha\|_\infty < \infty \}$. The set of all such $\phi$ is denoted by $L^{\infty}(\hat{G})$. It is a C*-algebra under the norm $\|\phi\|_\infty = \sup \{ \|\phi_\alpha\|_\infty : \alpha \in \hat{G} \}$ and coordinatewise operations (* denotes the operator adjoint).

Let $L^1(\hat{G}) = \{ \phi \in L^{\infty}(\hat{G}) : \|\phi\|_1 = \sum_{\alpha \in \delta} n_\alpha \|\phi_\alpha\|_1 < \infty \}$. Then $L^1(\hat{G})$ with the norm $\| \cdot \|_1$ is a Banach space and its dual may be identified with $L^{\infty}(\hat{G})$ under the pairing $\langle \phi, \psi \rangle = \sum_{\alpha \in \delta} n_\alpha \text{ Tr } (\phi_\alpha \psi_\alpha)$ (\phi \in L^1(\hat{G}), \psi \in L^{\infty}(\hat{G}))$. Let $\mu M(G)$, the measure algebra of $G$, then the Fourier transform of $\mu$, $\hat{\mu}$, is the function $\alpha \mapsto \hat{\mu}_\alpha = \int_\alpha T_\alpha(x^{-1}) \, d\mu(x)$ ($\alpha \in \hat{G}$), and $\hat{\mu} \in L^{\infty}(\hat{G})$ with $\| \hat{\mu} \|_\infty \leq \| \mu \|$. If $f \in C(G)$ (the continuous functions on $G$), then $\hat{f}_\alpha = \int_\alpha T_\alpha(x^{-1}) \, f(x) \, dm_\alpha(x)$ ($\alpha \in \hat{G}$).

We will now define $A(G)$, the Fourier algebra of $G$. Let

$$A(G) = \{ f \in C(G) : \hat{f} \in L^1(\hat{G}) \},$$

then $A(G)$ is in fact isomorphic to $L^1(\hat{G})$, since for $\phi \in L^1(\hat{G})$ the function $f(x) = \sum_{\alpha \in \delta} n_\alpha \text{ Tr } (\phi_\alpha T_\alpha(x))$ ($x \in G$) is continuous and $\hat{f} = \phi$. Put $\|f\|_A = \|\hat{f}\|_1$. Further
\( A(G) \) is a (commutative) Banach algebra under the pointwise operations on \( G \); and its dual is \( \mathcal{L}^\omega(\hat{G}) \), under the pairing

\[
\langle f, \phi \rangle = \sum_a n_a \text{Tr} (\hat{f}_a \phi_a) \quad (f \in A(G), \phi \in \mathcal{L}^\omega(\hat{G}));
\]

for proofs see [3, p. 93].

**Definition 1.1.** For \( \phi \in \mathcal{L}^\omega(\hat{G}) \) define the carrier of \( \phi \), \( \text{cr} \phi = \{ \alpha \in \hat{G} : \phi_\alpha \neq 0 \} \). For \( E \subseteq \hat{G} \), let \( \mathcal{L}^\omega(E) = \{ \phi \in \mathcal{L}^\omega(\hat{G}) : \text{cr} \phi \subseteq E \} \), and let \( A(E) = \{ f \in A(G) : \text{cr} f \subseteq E \} \). In fact \( A(E) \) is the closed span of \( \{ V_\alpha : \alpha \in E \} \).

**Proposition 1.2.** The spaces \( A(E) \), \( E \subseteq \hat{G} \), are exactly the closed subspaces of \( A(G) \) which are invariant under left and right translation by \( G \). The dual of \( A(E) \) is \( \mathcal{Y}_0(E) \).

**Proposition 1.3.** Let \( E, F \subseteq \hat{G} \), then the closed linear span of

\[
\{ fg : f \in A(E), g \in A(F) \}
\]

is equal to \( A(E \otimes F) \).

**Corollary 1.4.** For \( E \subseteq \hat{G} \), \( A(E) \) is a subalgebra of \( A(G) \) if and only if \( E \) is a subhypergroup. Further \( A(E) \) is a conjugate-closed \( (f \mapsto \bar{f}) \) subalgebra of \( A(G) \) if and only if \( E \) is a normal subhypergroup, and in that case

\[
A(E) = \{ f \in A(G) : f(h_1 x h_2) = f(x), \text{for all } h_1, h_2 \in E^\perp, x \in G \},
\]

the functions in \( A(E) \) constant on cosets of \( E^\perp \), a closed normal subgroup of \( G \).

**Corollary 1.5.** If \( E \) is a finite subhypergroup of \( \hat{G} \) then \( E \) is normal.

**Proof.** In fact \( A(E) \) is a finite dimensional subalgebra of a conjugate closed algebra \( A(G) \) and is thus itself conjugate-closed (since the maximal ideal space of \( A(E) \) is a finite set). \( \square \)

**Remark 1.6.** The Fourier algebra of a compact group \( G \) is the subject of [3, Chapter 8]. Helgason [7] constructed the duality between normal subhypergroups of \( \hat{G} \) and closed normal subgroups of \( G \). Translation-invariant uniformly closed linear subspaces of \( C(G) \) are discussed by Rider in [9]. What is observed by Iltis [8], that finite subhypergroups are normal, is implicit in Rider [9, p. 980].

2. Restrictions of representations to normal subgroups. In this section, \( G \) denotes a compact group, and \( H \) denotes a closed normal subgroup of \( G \). Define \( \hat{H} \) similarly to \( \hat{G} \), and denote the character of \( \gamma \in \hat{H} \) by \( \xi_\gamma \), and let \( T_\gamma^H \in \gamma \). Denote the normalized Haar measure of \( H \) by \( m_H \). There exists a natural homomorphism of \( G \) into the group of automorphisms of \( H \); namely, for \( x \in G \), let \( S_x h = x h x^{-1} \) (\( h \in H \)), then \( S_x \) is an automorphism of \( H \). Each \( S_x \) induces a permutation \( \hat{x} \) on \( \hat{H} \) such that \( \xi_\gamma(h) = \xi_{\gamma \gamma}(S_x h) \) (\( h \in H \)). Now define an equivalence relation on \( \hat{H} \) by \( \gamma_1 \sim \gamma_2 \) if and only if \( \gamma_2 = \gamma_1 \gamma_1^{-1} \) for some \( x \in G \), \( \gamma_1, \gamma_2 \in \hat{H} \). Denote the set of such equivalence classes by \( \hat{H} \). Let \( \alpha \in \hat{G} \) then \( T_\alpha \big| H \) is a continuous unitary representation of \( H \) and thus decomposes:

\[
T_\alpha \big| H = \sum_{\gamma \in \hat{H}} a_\gamma T_\gamma^H
\]
where the $a_\gamma$'s are nonnegative integers and only finitely many are nonzero.

**Theorem 2.1.** Let $\alpha \in \hat{G}$, then there is a positive integer $N_\alpha$ and a class $\Gamma_\alpha \in \hat{H}$ such that

$$\chi_\alpha | H = N_\alpha \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma.$$ 

Further each equivalence class in $\hat{H}$ is finite.

**Proof.** This theorem is nothing but the compact group analogue to the well-known finite groups result (see e.g. [1, p. 278]). We sketch an argument. Write $\chi_\alpha | H = \sum c_\gamma \xi_\gamma$. Now $\chi_\alpha | H$ is invariant under each $S_x, x \in G$, thus if $\gamma_1 \sim \gamma_2$ then $c_{\gamma_1} = c_{\gamma_2}$. It remains to show that if $c_{\gamma_1} \neq 0$ and $c_{\gamma_2} \neq 0$ then $\gamma_1 \sim \gamma_2$. Now $d \mu = \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma dm_H$ is a central measure in $M(G)$, and for any $\alpha \in \hat{G}$, $\mu$ is orthogonal to either all or none of the diagonal entry functions. If $\gamma' \notin \Gamma_\alpha$, then

$$\int_H \xi_{\gamma'} \left( \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma \right) dm_H = 0.$$

Thus the class $\Gamma_\alpha \in \hat{H}$ is uniquely determined and is evidently finite. However any $\gamma \in \hat{H}$ appears in the restriction of some $\alpha \in \hat{G}$ (induced representation argument) and thus any class in $\hat{H}$ is finite. □

**Corollary 2.2.** For any $\alpha, \beta \in \hat{G}$, either $n_{\beta} \chi_\alpha | H = n_\alpha \chi_\beta | H$ or $\int_H \chi_\alpha \bar{\chi}_\beta dm_H = 0$.

**Proof.** For $\alpha, \beta \in \hat{G}$, if $\Gamma_\alpha = \Gamma_\beta$ then $n_\alpha / N_\alpha = \sum_{\gamma \in \Gamma_\alpha} \chi_\gamma(e) = n_\beta / N_\beta$ ($e$ is identity in $G$). If $\Gamma_\alpha \neq \Gamma_\beta$, then $\int_H \chi_\alpha \bar{\chi}_\beta dm_H = 0$ by the orthogonality relations for characters of $H$. □

**Remark 2.3.** Rider uses this corollary in [10].

**Remark 2.4.** For $\alpha \in \hat{G}$, $\alpha \in H^\perp$ if and only if $\Gamma_\alpha = \{1\}$ ({$1$} denotes the trivial representation $H \to \{1\}$), and in this case, $\chi_\alpha | H = n_\alpha = N_\alpha$.

**Theorem 2.5.** For $\alpha, \beta \in \hat{G}$, $\alpha \in \beta \otimes H^\perp$ if and only if $\Gamma_\alpha = \Gamma_\beta$. Thus $\hat{G}$ is split into disjoint hypercosets, and these hypercosets are indexed by $\hat{H}$.

**Proof.** Let $\alpha \in \beta \otimes H^\perp$, then there exists some $\delta \in H^\perp$ such that $M_{\beta \delta}(\alpha) \neq 0$, that is $\chi_\delta \chi_\alpha = \chi_\alpha + \phi$, where $\phi$ is some nonnegative integer combination of characters. Now restrict to $H$ to obtain

$$n_\delta N_\beta \sum_{\gamma \in \Gamma_\beta} \xi_\gamma = N_\alpha \sum_{\gamma \in \Gamma_\alpha} \xi_\gamma + \phi | H,$$

thus $\Gamma_\alpha \subset \Gamma_\beta$, hence $\Gamma_\alpha = \Gamma_\beta$.

Conversely if $\Gamma_\alpha = \Gamma_\beta$, then $(\chi_\delta | H)(\chi_\beta | H) = 1 + \phi$ (some $\phi$ as above). (This follows from the relation $M_{\gamma}(\{1\}) = \int_H | \xi_\gamma |^2 dm_H = 1$ for any $\gamma \in \hat{H}$.) This implies that $(\alpha \otimes \beta) \cap H^\perp \neq \emptyset$. Thus there is a $\delta \in H^\perp$ such that $M_{\alpha \delta}(\delta) \neq 0$, but

$$M_{\alpha \delta}(\delta) = \int_G \chi_\delta \bar{\chi}_\delta dm_G = \int_G \bar{\chi}_\delta \chi_\delta dm_G = M_{\delta \beta}(\alpha)$$

and so $\alpha \in \beta \otimes H^\perp$. □
Observe that the Fourier algebra of $G/H$ is isomorphic to a closed subalgebra of $A(G)$, namely $A(H^\perp)$ (which we will denote by $A_H$), since $(G/H)^\perp$ may be identified with $H^\perp$. We will now decompose $A(G)$ into a direct sum of $A_H$-modules.

**Theorem 2.6.** To each $\Gamma \in \hat{H}$ there corresponds a closed subspace $A_\Gamma$ of $A(G)$ which is also an $A_H$-module. Further each $f \in A(G)$ has a unique decomposition $f = \sum_{\Gamma \in \hat{H}} f_{\Gamma}$, where $f_{\Gamma} \in A_{\Gamma}$ and $\|f\|_A = \sum_{\Gamma \in \hat{H}} \|f_{\Gamma}\|_A$. Also the $A_\Gamma$'s are the minimal closed left and right translation invariant $A_H$-submodules of $A(G)$.

**Proof.** For $\Gamma \in \hat{H}$, let $E_\Gamma = \{x \in G : \Gamma_x = \Gamma\}$, that is, the hypercoset of $H$ corresponding to $\Gamma$. Then put $A_\Gamma = A(E_\Gamma)$. Clearly $\hat{G}$ is the disjoint union of $\{E_\Gamma\}_{\Gamma \in \hat{H}}$, so the decomposition of $A(G)$ follows from the obvious decomposition of $L^1(\hat{G})$

Let $\Gamma \in \hat{H}$ and choose $x \in E_\Gamma$ then $E_{\Gamma} = \alpha \otimes H^\perp$, so that $E_{\Gamma} \otimes H^\perp = E_\Gamma$ and thus, by Proposition 1.3, $A_{H^\perp} \subset A_{\Gamma}$. So $A_{\Gamma}$ is a closed $A_H$-submodule of $A(G)$. If a nontrivial closed left and right translation invariant $A_H$-module is contained in $A_{\Gamma}$, then it is determined by some nonempty subset $F \subset E_\Gamma$. But if $\alpha \in F$ and $\beta \in H^\perp$ then $\alpha \otimes \beta \in F$, thus $F$ is a hypercoset, hence equals $E_\Gamma$. □

**Remark 2.7.** If $G$ is abelian then each $A_\Gamma$ has a single generator (in the algebraic as well as the topological sense) over $A_H$. In the general case for $\Gamma \in \hat{H}$ and some $\alpha \in E_\Gamma$, the functions $\{T_{\alpha ij} : 1 \leq i,j \leq n\}$ generate $A_\Gamma$ topologically, but it is not clear that they do so algebraically. However the following is true.

**Theorem 2.8.** Let $\Gamma \in \hat{H}$, then $A_\Gamma$ is a finitely generated $A_H$-module, that is there exists $g_1,\ldots,g_m \in A_{\Gamma}$ (some $m < \infty$) so that each $f \in A_\Gamma$ may be written as $f = \sum_{i=1}^{m} k_i g_i$, with $k_i \in A_H$. Further there exists a constant $M < \infty$, such that the functions $k_i$ may be chosen with $\|k_i\|_A \leq M \|f\|_A$.

**Proof.** In a paper of Dunkl [2] the following is shown: let $\tau$ be a continuous unitary representation of $H$ on a finite dimensional space $V$, and let $A(G, V)$ be the space of $V$-valued functions on $G$ with each coordinate function in $A(G)$. Define $M(\tau) = \{f \in A(G, V) ; f(hx) = \tau(h)f(x) \text{ for all } h \in H, x \in G\}$, denoted $A(\tau)$ in [2]. Then $M(\tau)$ is a finitely generated (algebraically) $A_H$-module.

We now point out the applicability of this theorem to the present situation. Pick $\alpha \in \Gamma$, and let $V = V_\alpha H$. Recall $V_\alpha = \text{Sp}\{T_{\alpha ij} : 1 \leq i,j \leq n\}$ so that $V$ is a finite dimensional space of continuous functions on $H$, and is in fact the left and right translation invariant (by $H$) space generated by $\{\xi_\gamma : \gamma \in \Gamma\}$. This shows that $V$ depends only on $\Gamma$, that for any $f \in A_{\Gamma}$, $f|H \in V$, and finally that $V$ is invariant under each $S_x, x \in G$ (that is, if $g \in V, x \in G$, then the function $h \mapsto g(xh^{-1})$ is in $V(h \in H)$). Observe that a continuous unitary representation $\tau$ of $H$ is realized on $V$, namely right translation, with the inner product on $V$ given by $(f,g)_H = \int_H \overline{f(h)} g dm_H (f,g \in V)$, and $\tau(h)f(h) = f(h_1h)$ ($f \in V, h, h_1 \in H$).

We claim that $M(\tau) = A_{\Gamma}$, in fact if $f \in A_{\Gamma}$ then assign to each $x \in G$ the function $f(x, \cdot) : h_1 \mapsto f(h_1x) = (R(x)f)(h_1)$. Now $A_{\Gamma}$ is invariant under the right translation
R(x) so R(x)f|H ∈ V, thus f(x, · ) ∈ V. Further for x ∈ G, h ∈ H, f(hx, h_1) = f(h_1hx) = f(x, h), that is, f(hx, · ) = τ(h)f(x, · ). Finally to check the coordinate functions of f(x, · ) let g ∈ V and consider the function x ↦ (f(x, · ), g) = \int_H f(hx)g(h)dm_H = μ * f(x), where μ is the measure (g(h^{-1}))-1dm_H(h), and so μ * f ∈ A(G). Conversely, if f ∈ M(τ), so f is of the form f(x, h), with f(x, · ) ∈ V, then put f(x) = f(x, e). Thus f ∈ A(G) (by finite dimensionality of V, point evaluation is a bounded linear functional). Further for each x ∈ G let g = R(x)f|H, then g(h) = f(hx) = f(hx, e) = τ(h)f(x, e) = f(x, h) so the function g ∈ V, thus f ∈ A_r. Hence A_r = M(τ) and thus there exist generators g_1, · · · , g_m ∈ A_r (some m < ∞).

Now consider the bounded linear map T: A_r × A_r × · · · × A_r (m copies) → A_r defined by T(k_1, · · · , k_m) = \sum_{i=1}^m k_i g_i. By the above paragraph T is onto and so by the open mapping theorem there exists M < ∞ such that \{T(k_1, · · · , k_m) : \|k_i\|_A ≤ M\} ⊃ \{f ∈ A_r : \|f\|_A ≤ 1\}. □

3. Homomorphisms. Let π be a continuous homomorphism of a compact group G into a compact group K, and let H be the kernel of π. Then π induces the map π_1: C(K) → C(G), given by π_1f(x) = f(πx), f ∈ C(K), x ∈ G. The adjoint of π_1, denoted by π^*, takes M(G) into M(K). Further π_1 maps A(K) into A(G), since A(K) is spanned by the continuous positive definite functions and these are preserved by π_1. Also π_1|A(K) is a bounded operator on A(K) since each f ∈ A(K) is a sum f = f_1 + f_2 + i(f_3 - f_4), f_i positive definite and \sum_{i=1}^4 |f_i(e)| ≤ 2\|f\|_A. Finally the adjoint of π_1|A(K) is a bounded map \hat{π} : L^∞(G) into L^∞(K). Let M(\hat{G}), M(\hat{K}) be the closures of M(G) ⊂ M(K) in L^∞(G) and L^∞(K) respectively.

PROPOSITION 3.1. \hat{π}M(\hat{G}) ⊂ M(\hat{K}).

Proof. Let μ ∈ M(G), then \hat{π}μ satisfies the following: ⟨f, \hat{π}μ⟩ = \int_G f(x^{-1}) dμ(x), f ∈ A(G). Now let g ∈ A(K), then

⟨g, \hat{π}μ⟩ = ⟨π_1g, \hat{μ}⟩ = \int_G g(πx^{-1}) dμ(x) = \int_K g(k^{-1}) dπ^*μ(k) = ⟨g, (π^*μ)⟩.

Thus \hat{π}μ = (π^*μ) ∈ M(K). The continuity of \hat{π} finishes the proof. □

Observe that π factors into G → G/H → K, where G/H is identified with a closed subgroup of K. Further M(G/H) is identified with a closed subalgebra of M(G), namely m_H * M(G) (note m_H is an idempotent, see [3, Chapter 9]). Also L^∞((G/H) \cap L^∞(H^1)) and M((G/H) \cap L^∞(H^1)) (since m_H is the projection of L^∞(G) onto L^∞(H^1)).

Finally \hat{π} takes M(\hat{G}) onto M(\hat{K}), or L^∞(\hat{G}) onto L^∞(\hat{K}) if and only if π maps G onto K, for otherwise πG is a proper closed subgroup of K, and φ ∈ \hat{π}L^∞(\hat{G}) if and only if spt φ ⊂ πG (where the support of φ, spt φ, is the least compact subset E ⊂ K with the property that f ∈ A(K), f = 0 on a neighborhood of E implies ⟨f, φ⟩ = 0).
Now we investigate the effect of \( \hat{\pi} \) on \( W(\hat{G}) \), the weakly almost periodic (w.a.p.) elements of \( \mathcal{L}^\omega(\hat{G}) \). We state some appropriate definitions and results from our previous paper [6].

**Proposition 3.2.** \( \mathcal{L}^\omega(\hat{G}) \) is an \( A(G) \)-module. The action is defined by \( \langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle \) (\( f, g \in A(G) \), \( \phi \in \mathcal{L}^\omega(\hat{G}) \)), and \( \| f \cdot \phi \|_\infty \leq \| f \|_A \| \phi \|_\infty \). Further \( \text{cr} (f \cdot \phi) \subseteq \text{cr} f \otimes \text{cr} \phi \).

**Definition 3.3.** For \( \phi \in \mathcal{L}^\omega(\hat{G}) \), one says that \( \phi \) is weakly almost periodic if the map \( f \mapsto f \cdot \phi \) is a weakly compact operator of \( A(G) \) into \( \mathcal{L}^\omega(\hat{G}) \) (\( f \in A(G) \)). The set of all such \( \phi \) is denoted by \( W(\hat{G}) \).

**Definition 3.4.** Let \( B = \{ f \in A(G) : \| f \|_A \leq 1 \} \). For \( \alpha \in \hat{G} \), let \( B_\alpha = B \cap V_\alpha \). Let \( E = \bigcup_{\alpha \in \hat{G}} B_\alpha \).

Some properties of \( W(\hat{G}) \) (see [6]):

1. \( W(\hat{G}) \) is a closed submodule of \( \mathcal{L}^\omega(\hat{G}) \).
2. For \( \phi \in \mathcal{L}^\omega(\hat{G}) \) to be in \( W(\hat{G}) \) it is necessary and sufficient that \( \{ f_n \cdot \phi \} \) have a weakly convergent subsequence for any sequence \( \{ f_n \} \subseteq E \) (also true if \( E \) is replaced by \( B \)).

**Theorem 3.5.** Let \( \pi \) be a continuous homomorphism of \( G \) into \( K \) (compact groups). Then \( \hat{\pi} W(\hat{G}) \subseteq W(K) \).

**Proof.** Suppose \( \phi \in W(\hat{G}) \), and \( \{ f_n \} \) is a bounded sequence in \( A(K) \). Then \( \{ \pi_1 f_n \} \) is a bounded sequence in \( A(G) \), and there exists a subsequence such that \( (\pi_1 f_n) \cdot \phi \) converges weakly to \( \psi \in \mathcal{L}^\omega(\hat{G}) \). But \( \hat{\pi} ((\pi_1 f_n) \cdot \phi) = f_n \cdot (\hat{\pi} \phi) \), so \( f_n \cdot \hat{\pi} \phi \) converges weakly to \( \hat{\pi} \psi \in \mathcal{L}^\omega(\hat{K}) \) (for \( \hat{\pi} \), being strongly continuous, is weakly continuous). Hence \( \hat{\pi} \phi \in W(\hat{K}) \). \( \square \)

Henceforth we assume \( \pi \) is onto \( K \) so we identify \( K \) with \( H^\perp \), and \( \mathcal{L}^\omega(\hat{K}) \) with \( \mathcal{L}^\omega(H^\perp) \). We have just seen that the restriction map \( \hat{\pi} : \mathcal{L}^\omega(\hat{G}) \rightarrow \mathcal{L}^\omega(H^\perp) \) takes \( W(\hat{G}) \) into \( W(\hat{K}) \). We will now show that in fact \( \hat{\pi} W(\hat{G}) \cap \mathcal{L}^\omega(H^\perp) = W(\hat{K}) \).

**Definition 3.6.** Let \( \{ \phi_n \} \) be a sequence in \( \mathcal{L}^\omega(\hat{G}) \). Say \( \phi_n \rightharpoonup \phi \) \( \in \mathcal{L}^\omega(\hat{G}) \) quasi-uniformly if \( (\phi_n)_a \rightarrow \phi_a \) for each \( \alpha \in \hat{G} \), and for each \( \varepsilon > 0 \), \( N = 1, 2, 3, \ldots \), there exist integers \( m_1, \ldots, m_k \geq N \), such that \( \min_{1 \leq i \leq k} \| (\phi_{m_i})_a - \phi_a \|_\infty < \varepsilon \) for each \( \alpha \in \hat{G} \).

**Theorem 3.7 [6].** Let \( \{ \phi_n \} \subseteq \mathcal{L}^\omega(\hat{G}) \). Then \( \phi_n \rightharpoonup \phi \in \mathcal{L}^\omega(\hat{G}) \) weakly if and only if \( \sup_n \| \phi_n \|_\infty < \infty \), and every subsequence of \( \{ \phi_n \} \) converges quasi-uniformly to \( \phi \).

**Theorem 3.8.** Let \( \phi \in W(\hat{K}) \), that is, \( \phi \in \mathcal{L}^\omega(H^\perp) \), and for each bounded sequence, \( \{ f_n \} \subseteq A(K) = A_H \) (see previous section), \( \{ f_n \cdot \phi \} \) has a weakly convergent subsequence. Then \( \phi \in W(\hat{G}) \) (note \( \phi_a = 0 \) for \( \alpha \notin H^\perp \)).

**Proof.** Let \( \{ f_n \} \subseteq E = \bigcup_a B_a \), with \( f_n \in B_{a_n} \), \( n = 1, 2, 3, \ldots \). We must show that \( \{ f_n \cdot \phi \} \) has a weakly convergent subsequence. There are two possibilities for \( \{ a_n \} \):

1. There are infinitely many distinct cosets \( a_n \otimes H^\perp \). That is, there exists a subsequence \( f_{n_j} \) such that the sets \( \text{cr} (f_{n_j} \cdot \phi) \subseteq \bar{a}_{n_j} \otimes H^\perp \) are all disjoint. Then \( f_{n_j} \cdot \phi \rightharpoonup 0 \) weakly by Theorem 3.7.
(2) Infinitely many $\alpha_n \in \alpha \otimes H^\perp$, some $\alpha \in \hat{G}$. Thus there is a bounded subsequence $f_{n_j}$ in $A_\Gamma$, where $\Gamma = \Gamma_\alpha$ (recall Theorem 2.6). By Theorem 2.8, there exist $g_1, \ldots, g_m \in A_\Gamma$ and functions $h_{ij} \in A_H$, and $M < \infty$, such that $f_{n_j} = \sum_{i=1}^m h_{ij} g_i$, and $\|h_{ij}\|_A \leq M$, all $i, j$. By successively extracting subsequences from $\{h_{11}\}, \{h_{22}\}, \ldots, \{h_{mj}\}$ and reindexing, we obtain $\psi_1, \ldots, \psi_m \in L^\infty(H^\perp)$ such that $h_{ij} \cdot \psi_i$ weakly, $i = 1, \ldots, m$. The map $\psi \mapsto g_i \cdot \psi$ on $L^\infty$ is strongly, hence weakly continuous, thus

$$f_{n_j} \cdot \phi = \sum_{i=1}^m g_i \cdot (h_{ij} \cdot \phi) \rightharpoonup \sum_{i=1}^m g_i \cdot \psi_i \text{ weakly.} \qed$$

**Corollary 3.9.** If $W(\hat{K}) \neq L^\infty(\hat{K})$ then $W(\hat{G}) \neq L^\infty(\hat{G})$. If $M(\hat{K}) \neq W(\hat{K})$ then $M(\hat{G}) \neq W(\hat{G})$. (Recall from [6] that $M(\hat{G}) \subset W(\hat{G})$.)

**Corollary 3.10.** If $G$ has an infinite abelian image, then $M(G) \neq W(\hat{G}) \neq L^\infty(\hat{G})$.

**Proof.** If $K$ is an infinite compact abelian group, then $M(\hat{K}) \neq W(\hat{K}) \neq L^\infty(\hat{K})$ (see [3, Chapter 4] and [6]). $\square$

**Remark 3.11.** In [4] we show that $W(\hat{G}) \neq L^\infty(\hat{G})$ for any infinite compact group $G$. In [5] we show that $M(\hat{G}) \neq W(\hat{G})$ for any compact group $G$ which contains an infinite abelian subgroup.

**Bibliography**

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