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LOCALLY COMPACT SUBGROUPS OF THE SPECTRUM OF THE MEASURE ALGEBRA II

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The maximal ideal space Δ_G of the measure algebra $M(G)$ of a locally compact abelian group G is a compact commutative semitopological semigroup. In this paper we show that $c\ell\,\hat{G}$ the closure of $\hat{G}$, the dual of G, in Δ_G can contain maximal subgroups which are not locally compact. We have previously characterized the locally compact maximal subgroups of $c\ell\,\hat{G}$ as arising from locally compact topologies on G which are finer than the original topology.

We use the notation from our paper [1]. The following lemma is standard.

LEMMA 1. *Let Δ be a compact commutative semi-topological semigroup and $\{\varepsilon_\alpha\}$ a net of idempotents from Δ with $\alpha \leq \beta$ implying $\varepsilon_\alpha \times \varepsilon_\beta = \varepsilon_\alpha$. Then there exists an idempotent ε_∞ such that $\varepsilon_\alpha \overset{\alpha}{\to} \varepsilon_\infty$ and $\varepsilon_\alpha \times \varepsilon_\infty = \varepsilon_\alpha$ for all α.*

THEOREM 2. *Let G be a locally compact abelian group. Suppose in $c\ell\,\hat{G}$, there exists a net of idempotents $\{\varepsilon_\alpha\}$ converging to an idempotent ε_∞ and*

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$\varepsilon_\infty \times \varepsilon_\alpha = \varepsilon_\alpha$ _for all_ α. _Suppose further_ $\varepsilon_\alpha \neq \varepsilon_\infty$ _for all_ α. _Then the maximal subgroup_ $H(\varepsilon_\infty)$ _of_ $c\ell\,\hat{G}$ _containing_ ε_∞ _is not locally compact_.

Proof. Suppose $H(\varepsilon_\infty)$ is locally compact, then $H(\varepsilon_\infty)$ is the dual of G_∞, G with a finer locally compact topology (see [1, Corollary 11]). Then $\varepsilon_\infty \times c\ell\,\hat{G}$ is isomorphic to $c\ell\,\hat{G}_\infty$, the closure of $\hat{G}_\infty$ in the maximal ideal space of $M(G_\infty)$. The Gelfand transforms of elements of $L^1(G_\infty)$ are zero off $\hat{G}_\infty$ in $c\ell\,\hat{G}_\infty$. Thus the Gelfand transforms of elements of $L^1(G_\infty)$, as a subalgebra of $M(G)$, are zero off $H(\varepsilon_\infty)$ in $\varepsilon_\infty \times c\ell\,\hat{G}$. In particular, $\tilde{\mu}(\varepsilon_\alpha \times \gamma) = 0$ for $\mu \in L^1(G_\infty)$, $\gamma \in \hat{G}$, all α (where $\tilde{\mu}$ is the $M(G)$-Gelfand transform of μ), since $\varepsilon_\alpha \in (\varepsilon_\infty \times c\ell\,\hat{G}) \setminus H(\varepsilon_\infty)$. Let $\gamma \in \hat{G}$ and $\mu \in L^1(G_\infty)$, then $\tilde{\mu}(\gamma \times \varepsilon_\infty) = \lim_\alpha \tilde{\mu}(\gamma \times \varepsilon_\alpha) = 0$. But $\varepsilon_\infty \times \hat{G}$ is dense in $H(\varepsilon_\infty)$, and so $\tilde{\mu} = 0$ on $H(\varepsilon_\infty)$, a contradiction. □

We remark that for $H(\varepsilon)$ locally compact, $H(\varepsilon)$ is the maximal group containing ε both in $c\ell\,\hat{G}$ and Δ_G, (see [1, Corollary 11]).

<u>EXAMPLE</u>. Let $G = T^\omega = \Pi_{j=1}^\infty T_j$ (the complete direct product of a countable number of copies of the circle group T with the product topology $\mathcal{T}$). Let $G_n = T^n \times \Pi_{j=n+1}^\infty (T_d)_j$ (T_d denotes T with the discrete topology and $\Pi_{j=n+1}^\infty (T_d)_j$ has the discrete topology). Let $\mathcal{T}_n$ denote the (locally compact) topology on G_n, and let $\mathcal{K}_n$ denote the collection of compact subsets of $\mathcal{T}_n$. Each $\mathcal{T}_n$ is a locally compact topology for G finer than $\mathcal{T}$, and so each $\mathcal{K}_n$ generates a Raikov projection P_n such that

$P_n M(G) = M(G_n)$ and $P_n = E_{\varepsilon_n}$ for $\varepsilon_n \in c\ell\, \hat{G}$, (see [1, Theorem 4]). Since $\mathcal{Y}_1 \supset \mathcal{Y}_2 \supset \dots$, thus $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$, and so $M(G_1) \subset M(G_2) \subset \dots$ which implies $\varepsilon_n \times \varepsilon_m = \varepsilon_n$ for $n \leq m$. By Lemma 1, there exists an idempotent ε_∞ such that $\varepsilon_n \overset{n}{\to} \varepsilon_\infty$ and $\varepsilon_n \times \varepsilon_\infty = \varepsilon_n$. Now $E_{\varepsilon_\infty} M(G) \supset \bigcup_{n=1}^{\infty} M(G_n)$ and so $\varepsilon_\infty \neq \varepsilon_n$, $n=1,2,\dots$. Theorem 2 now yields the following theorem.

THEOREM 3. *Let $G = T^\omega$. Then $c\ell\, \hat{G}$ contains a maximal subgroup which is not locally compact.*

For G a nondiscrete LCA group, a result of B. Johnson [2, Chapter 2] shows that $c\ell\, \hat{G}$ is not the union of its maximal subgroups. In particular, Johnson showed the existence of a probability measure $\mu \in M(G)$, and an element $\pi \in c\ell\, \hat{G}$ with the associated generalized character $|f_\pi^\mu| = c$, $0 < c < 1$; and thus π is not in any subgroup of $c\ell\, \hat{G}$.

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