LOCALLY COMPACT SUBGROUPS
OF THE SPECTRUM OF THE MEASURE ALGEBRA
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The maximal ideal space $\Delta_G$ of the measure algebra $M(G)$ of a locally compact abelian group $G$ is a compact commutative semitopological semigroup. In this paper a class of locally compact subgroups of the closure of $\hat{G}$, the dual group of $G$, in $\Delta_G$ is characterized. Each such group is the dual of the abstract group $G$ with some stronger locally compact topology than that of $G$. There is no more than one such group about any idempotent in the closure of $\hat{G}$. In a previous paper the authors showed that every stronger locally compact topology on $G$ determines an idempotent in the closure of $\hat{G}$. In other words, an exact description is given of those idempotents in the closure of $\hat{G}$ which are contained in locally compact maximal subgroups of $\Delta_G$.

NOTATION. Denote the dual of an LCA (locally compact abelian) group $G$ by $\hat{G}$. For LCA groups $G$ and $H$, and a morphism (continuous homomorphism) $\phi: G \to H$, there exists the dual morphism $\hat{\phi}: \hat{H} \to \hat{G}$, defined by $\hat{\phi}\gamma(x) = \gamma(\phi x)$ for $x \in G$, $\gamma \in \hat{H}$. For $G$ an LCA group, $M(G)$ denotes the measure algebra, and $\Delta_G$ denotes the maximal ideal space of $M(G)$. As

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usual, we identify $\Delta_G$ with the space of nonzero multiplicative linear functionals on $M(G)$ furnished with the Gelfand topology. Let $M_d(G)$ be the closed subalgebra of discrete measures and $L^1(G)$ the ideal of absolutely continuous measures. For $\mu \in M(G)$, $\hat{\mu}$ is the Fourier-Stieltjes transform given by $\hat{\mu}(\gamma) = \int_G \gamma d\mu$, and $\hat{\mu}$ is the Gelfand transform, a continuous function on $\Delta_G$. We will identify $\hat{G}$ with a subset of $\Delta_G$, so that $\hat{\mu}(\gamma) = \hat{\nu}(\gamma)$ for $\mu \in M(G)$, $\gamma \in G$. For $x \in G$, let $\delta_x$ be the unit mass at $x$. Generally we will use additive notation for $G$, and multiplicative notation for $\hat{G}$.

Henceforth $G$ denotes a fixed LCA group.

We now recall some facts about Raikov systems.

**DEFINITION 1.** A nontrivial class $\mathcal{F}$ of $\sigma$-compact subsets of $G$ is called a Raikov system if

1) $A \in \mathcal{F}$, $B$ a $\sigma$-compact subset of $G$ and $B \subseteq A$ imply $B \in \mathcal{F}$,

2) if $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$ then $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$,

3) $A, B \in \mathcal{F}$ implies $A + B \in \mathcal{F}$, and

4) $A \in \mathcal{F}$ and $x \in G$ imply $A + x \in \mathcal{F}$.

**THEOREM [5].** Given a Raikov system $\mathcal{F}$ there exists a unique bounded algebraic projection $P$ on $M(G)$ such that $PM(G) = \{\mu \in M(G): \mu \text{ is carried by some } A \in \mathcal{F}\}$ and the kernel of $P$, an ideal, is the set $\{\mu \in M(G): |\mu|A = 0 \text{ for all } A \in \mathcal{F}\}$.

We will call such a projection a Raikov projection.

**DEFINITION 2.** Let $\phi: H \to G$ be a monomorphism. Let $G_H$ be the group $G$ topologized so that $\phi_H$
has the H-topology and is open in $G_H$. Note $G_H$ is LCA and the identity: $G_H \to G$ is continuous. Let $P$ be the Raikov projection associated with the Raikov system generated by the $G_H$-compact sets. We say $P$ is the Raikov projection induced by $\phi$.

**DEFINITION 3.** Let $\phi: H \to G$ be a morphism. Define $\phi^*: M(H) \to M(G)$, a homomorphism, by $\phi^*\mu(E) = \mu(\phi^{-1}E)$ for each Borel set $E \subseteq G$, $\mu \in M(H)$. Equivalently $(\phi^*\mu)(\gamma) = \hat{\mu}(\hat{\phi}\gamma)$, for $\gamma \in \hat{G}$. If $\phi$ is a monomorphism then $\phi^*$ is an isometric isomorphism of $M(H)$ into $M(G)$ (see [4] and [6]).

**PROPOSITION 1.** With the hypotheses of the two prior definitions, the range of $P$ may be identified with $M(G_H)$. If further, $\phi$ maps $H$ onto $G$, then $\phi^*$ is an isometric isomorphism of $M(H)$ onto $M(G_H)$. We now single out an important set of operators on $M(G)$.

**DEFINITION 4.** Say $T$ is an $R$-homomorphism of $M(G)$ if $T$ is a bounded nonzero homomorphism of $M(G)$ into itself, such that $(T\mu)|E = T(\mu|E)$ for $\mu \in M(G)$, $E$ Borel $\subseteq G$. That is, $T$ commutes with restriction to Borel sets. If $T$ is a projection and an $R$-homomorphism, then $T$ is called an $R$-projection.

**PROPOSITION 2.** If $T$ is an $R$-homomorphism, and $x \in G$, then $T\delta_x = \chi_T(x)\delta_x$, where $\chi_T$ is some character (not necessarily continuous) of $G$. If further $T$ is an $R$-projection, then $T\delta_x = \delta_x$.

Proof. For any $\mu \in M(G)$, $T\mu = T(\mu*\delta_o) = T\mu*T\delta_o$, but $T \neq 0$, so $T\delta_o \neq 0$. For any $x \in G$, $T\delta_x|\{x\}$ = $T\delta_x$, so there exists a complex number $\chi_T(x)$
such that $T \delta_x = \chi_T(x) \delta_x$. But since $T \delta_{x+y} = T(\delta_x \ast \delta_y) = T \delta_x \ast T \delta_y$, we have $\chi_T(x+y) = \chi_T(x) \chi_T(y)$, for all $x, y \in G$, and $\chi_T(0) \neq 0$ so $\chi_T$ is a character of $G$. If $T$ is an $R$-projection, then $(\chi_T)^2 = \chi_T$, so $T \delta_x = \delta_x$, all $x \in G$. □

**COROLLARY 3.** The set of $R$-homomorphisms is closed under composition, and is a semigroup with identity $I$ (where $I$ is the identity map on $M(G)$).

Proof. It suffices to observe, that $S \circ T(\delta_o) = S(\delta_o) = \delta_o$ so $S \circ T \neq 0$, for $R$-homomorphisms $S$ and $T$. □

We briefly sketch the details of the isomorphism between the semigroup $\mathcal{E}$ of $R$-homomorphisms and the compact space $\Delta_G$ (see [2, Chapter 1]). For each $\tau \in \Delta_G$ there exists a unique generalized character $\{f^\mu_\tau\}$, where $f^\mu_\tau \in L^\infty(\mu)$ for each $\mu$ of the form $\mu = \exp |\nu|$, $\nu \in M(G)$; and for each $\lambda \in L^1(\nu)$ (that is, the space of elements of $M(G)$ which are absolutely continuous with respect to $\nu$) we have the relation $\chi(\tau) = \int_G f^\mu_\tau d\lambda$. Observe that for each such $\mu$, that $\tau | L^1(\mu)$ is a bounded linear functional and thus determines an element $f^\mu_\tau$ of $L^\infty(\mu)$ with the required property. Note further that if $\nu_1$ and $\nu_2$ are both exponentials, and $\nu_1 \in L^1(\nu_2)$ then $f^{\nu_2}_\tau = f^{\nu_1}_\tau \nu_1$-almost everywhere. One can also prove that for $\mu = \exp |\nu|$, $\nu \in M(G)$, that $f^\mu_\tau(x+y) = f^\mu_\tau(x) f^\mu_\tau(y)$ for $(x,y) \in G \times G$, $\nu \times \nu$ - almost everywhere.

Now to each $\tau \in \Delta_G$ we associate the $R$-homomorphism $E_\tau$ defined by $E_\tau \lambda = f^\mu_\tau \lambda$ (where $\lambda \in M(G)$,
and $\mu = \exp |\lambda|$.

Conversely, if $E$ is an $R$-homomorphism, define $\tau \in \Delta_G$ by $\hat{\nu}(\tau) = (E\mu)^\wedge(1)$, $(\mu \in M(G))$, then it can be shown that $E = E_\tau$ (see [2, Chapter 1]. Now we give $\Delta_G$ the structure of a semigroup so that for $\sigma, \tau \in \Delta_G$, $\sigma \times \tau$ is described by the generalized character $f^\mu_\sigma f^\mu_\tau$; or equivalently, that $E_{\sigma \times \tau} = E_\sigma E_\tau$.

Further it can be shown that multiplication ($\times$) is separately continuous on $\Delta_G$, so $\Delta_G$ becomes a commutative compact semitopological semigroup with identity $1$, (for a general reference see [1]). The $R$-projections correspond to idempotents in $\Delta_G$. Note that $\hat{G}$ is embedded homeomorphically into $\Delta_G$, in fact, as a subgroup, since for $\gamma_1, \gamma_2 \in \hat{G}$ we have $\gamma_1 \times \gamma_2 = \gamma_1 \gamma_2$; the generalized character $f^\mu_\gamma$ for $\gamma$ is exactly $\gamma$ (as a function on $G$).

The space that will hold our interest is the closure of $\hat{G}$, $c\ell \hat{G}$, a compact subsemigroup of $\Delta_G$. For $B$ a set of symmetric maximal ideals of $\Delta_G$, $\overline{B}$ consists of all $\tau \in \Delta_G$ for which $|\hat{\nu}(\tau)| \leq \sup\{|\hat{\nu}(\pi)| : \pi \in B\}$; for if the inequality holds, $\tau$ can be extended to a multiplicatively linear functional on $C(\overline{B})$ (since $M(G)^\wedge$ $\overline{B}$ is sup-norm dense in $C(\overline{B})$ by the Stone-Weierstrass theorem), and so $\tau \in \overline{B}$. Thus for $\tau \in \Delta_G$, $\tau \in c\ell \hat{G}$ if and only if $|\nu(\tau)| \leq ||\hat{\nu}||_\infty$, for all $\mu \in M(G)$. An equivalent formulation is that $||(E_\tau \mu)^\wedge||_\infty \leq ||\hat{\mu}||_\infty$, $\mu \in M(G)$; and to see this, observe that $(E_\tau \mu)^\wedge(\gamma) = \int_\gamma f^\lambda_\tau d\mu = \hat{\nu}(\gamma \times \tau)$, for $\gamma \in \hat{G}$, $\mu \in M(G)$, $\lambda = \exp |\mu|$. Clearly $\sup\{|\hat{\nu}(\sigma) : \sigma \in c\ell \hat{G}\} = ||\hat{\nu}||_\infty$ and $\gamma \times \tau \in c\ell \hat{G}$ for all.
\( \gamma \in \hat{G} \) if and only if \( \tau \in c\ell \hat{G} \).

Observe that any Raikov projection is an R-projection. However, Yu. Šreider [8] has constructed an R-projection which is not a Raikov projection. In our paper [3] we proved the following:

**THEOREM 4.** Let \( \phi : H \to G \) be a monomorphism, and let \( P \) be the Raikov projection induced by \( \phi \). Then
\[
|| (P \mu) \hat{\gamma} ||_\infty \leq || \hat{\mu} ||_\infty.
\]
Thus \( P = E_\varepsilon \), for \( \varepsilon \in c\ell \hat{G} \).

We will describe how Theorem 4 gives the existence of LCA subgroups of \( c\ell \hat{G} \), and then characterize the LCA subgroups of \( c\ell \hat{G} \) which arise this way.

For further illustrations of the theory of R-projections we give the following:

**PROPOSITION 5.** If \( P \) is an R-projection and \( PL^1(G) \neq \{0\} \), then \( P = I \).

**Proof.** Write \( P = E_\varepsilon \), \( \varepsilon \) an idempotent in \( \Delta_G \).

Then for any \( \mu \in M(G) \), \( (P \mu) \hat{\gamma} = \hat{\mu}(\gamma \times \varepsilon) \), but if \( \mu \in L^1(G) \), then \( \hat{\mu} = 0 \) off \( \hat{G} \). (Note that \( \hat{G} \) is the spectrum of \( L^1(G) \); see [7, p. 7].) Hence if \( PL^1(G) \neq \{0\} \), there exists \( \gamma_1, \gamma_2 \in \hat{G} \) such that
\[
\gamma_1 \times \varepsilon = \gamma_2,
\]
but then \( \varepsilon = \gamma_1 \gamma_2 \in \hat{G} \), so \( \varepsilon = 1 \).

Recall from Proposition 2 that if \( P \) is an R-projection then \( P \) is the identity on \( M_d(G) \).

**EXAMPLE.** There exists an LCA group \( G \), and a bounded algebraic projection \( P \) on \( M(G) \), such that \( PL^1(G) = \{0\} \), \( P \) is the identity on \( M_d(G) \),
\[
|| (P \mu) \hat{\gamma} ||_\infty \leq || \hat{\mu} ||_\infty,
\]
all \( \mu \in M(G) \), but \( P \) is not an R-projection.

**Proof.** Let \( G \) be an LCA group having subgroups \( H \) and \( K \neq \{0\} \) such that \( H \) has its own LCA
topology, and $K$ is compact, $K \subseteq H$ and such that
the injection maps $K \to H \to G$ are continuous, and not
open. Let $P_K$ be the Raikov projection $M(G) \to M(G_K)$
induced by $K \to G$, and let $P_H$ be the Raikov projec-
tion $M(G) \to M(G_H)$ induced by $H \to G$. Let
$T: M(G) \to M(G)$ be defined by

$$T\mu = (P_H \mu) * m_K + (\delta - m_K) * P_K \mu, \text{ for } \mu \in M(G),$$

where $m_K$ is the normalized Haar measure on $K$. A
direct computation shows that $T$ is a projection on $M(G)$ (note that $P_H m_K = P_K m_K = m_K$).
The Fourier-Stieltjes transform of $T\mu$ is given
by

$$(T\mu)^\wedge(\gamma) = \begin{cases} (P_H \mu)^\wedge(\gamma), \gamma \in K^\perp \\ (P_K \mu)^\wedge(\gamma), \gamma \notin K^\perp \end{cases}$$

(where $K^\perp$ is the annihilator of $K$ in $\hat{G}$). Thus
$\| (T\mu)^\wedge \|_\infty \leq \| \hat{\mu} \|_\infty$, and $T$ is an algebraic projec-
tion, which is the identity on $M_d(G)$ and
$TL^1(G) = \{0\}$. To show that $T$ is not on R-projection
it suffices to show $T(\gamma \mu) \neq \gamma T\mu$, for some $\gamma \in \hat{G},$
$\mu \in M(G)$, (since R-projections commute with multiplica-
tion by bounded continuous functions). Let $\gamma \notin K^\perp,$
and let $\mu \in L^1(G_H)$ such that $\hat{\mu}(1) \neq 0$, then
$(T(\gamma \mu))^\wedge(\gamma) = (P_K (\gamma \mu))^\wedge(\gamma) = 0$ but $(\gamma T\mu)^\wedge(\gamma) = (T\mu)^\wedge(1) = (P_H \mu)^\wedge(1) = \hat{\mu}(1) \neq 0$. \hfill $\Box$

**Definition 5.** Let $\varepsilon$ be an idempotent in $c^0\hat{G}$. Then the maximal subgroup of $c^0\hat{G}$ containing $\varepsilon$, denoted by $H(\varepsilon)$, is the set $\{ \sigma \in c^0\hat{G}: \sigma \times \varepsilon = \sigma,$ and there exists $\tau \in c^0\hat{G}$ such that $\sigma \times \tau = \varepsilon \}$. It can be shown that $\sigma \in H(\varepsilon)$ implies $|f^\mu_\sigma| = f^\mu_\varepsilon$, for
all \( \mu = \exp |\lambda| \), \( \lambda \in M(G) \); and for \( \{ \sigma_{\alpha} \} \) a net in \( H(\varepsilon) \), \( \sigma_{\alpha} \rightarrow \sigma \in H(\varepsilon) \) (in the \( \Lambda^G \) topology) if and only if \( \int_\Gamma |f_{\sigma_{\alpha}}^{\mu} - f_{\sigma}^{\mu}| d\mu = 0 \) for all \( \mu = \exp |\lambda| \), \( \lambda \in M(G) \). Thus \( H(\varepsilon) \) has jointly continuous multiplication, and is thus a topological group.

**DEFINITION 6.** For \( \varepsilon \) an idempotent in \( \mathcal{c}_{\mathcal{L}} \hat{G} \), let \( \rho_{\varepsilon} \) be the continuous map \( \gamma \mapsto \gamma \times \varepsilon \) for \( \gamma \in \hat{G} \).

**PROPOSITION 6.** For \( \varepsilon \) an idempotent in \( \mathcal{c}_{\mathcal{L}} G \), \( \rho_{\varepsilon} \) is a continuous monomorphism of \( \hat{G} \) with dense range in \( H(\varepsilon) \). Further \( \mathcal{c}_{\mathcal{L}} H(\varepsilon) = \varepsilon \times \mathcal{c}_{\mathcal{L}} \hat{G} = \{ \sigma \in \mathcal{c}_{\mathcal{L}} \hat{G} : \sigma \times \varepsilon = \sigma \} \).

Proof. Since \( \varepsilon \) is an idempotent, \( \rho_{\varepsilon} \) is a continuous homomorphism. Now suppose for some \( \gamma \in \hat{G} \) that \( \varepsilon \times \gamma = \varepsilon \), then for each \( x \in G \), \( (\delta_x^{\varepsilon})^{\gamma}(\varepsilon \times \gamma) = (E_x^{\varepsilon})(\gamma) = (\delta_x^{\varepsilon})(\gamma) = \gamma(x) \) and \( (\delta_x^{\varepsilon})(\varepsilon \times \gamma) = (\delta_x^{\varepsilon})(\varepsilon) = 1 \); hence \( \gamma = 1 \) and \( \rho_{\varepsilon} \) is a monomorphism. Finally \( \varepsilon \times \hat{G} \subset H(\varepsilon) \subset \varepsilon \times \mathcal{c}_{\mathcal{L}} G \), and \( \mathcal{c}_{\mathcal{L}} (\varepsilon \times \hat{G}) = \varepsilon \times \mathcal{c}_{\mathcal{L}} \hat{G} \).

**PROPOSITION 7.** Let \( \varepsilon \) be an idempotent in \( \mathcal{c}_{\mathcal{L}} \hat{G} \), then there exists a morphism \( j : H(\varepsilon) \rightarrow \beta \hat{G} \), the Bohr compactification of \( \hat{G} \), which is identified with \( (G_\delta^\hat{G})^\wedge \), \( G_\delta \) is \( G \) with the discrete topology. Further \( j \circ \rho_{\varepsilon} = 1 \), the canonical injection of \( \hat{G} \) into \( \beta \hat{G} \); so \( j \) has dense range.

Proof. For each \( \sigma \in H(\varepsilon) \), define a character on \( G \) (hence an element of \( \beta \hat{G} \)) by \( j_\sigma(x) = (\delta_x^{\varepsilon})(\sigma) \), \( x \in G \); \( j_\sigma \) is a character by the argument of Proposition 2). Further if a net \( \{ \sigma_{\alpha} \} \subset H(\varepsilon) \) converges to \( \sigma \in H(\varepsilon) \), then for each \( x \), \( j_{\sigma_{\alpha}}(x) \rightarrow j_\sigma(x) \),
since \( j_{\sigma^\alpha}(x) = (\delta_x)^{\gamma}(\sigma^\alpha) \cdot (\delta_x)^{\gamma}(\sigma) \). Let \( \gamma \in \hat{G} \), \( x \in G \), then
\[ j_{\rho^\gamma}\varepsilon(x) = (\delta_x)^{\gamma}(\gamma \times \varepsilon) = (E_{\delta_x}^\gamma)^{\gamma}(\gamma) = (\delta_x)^{\gamma}(\gamma) = \gamma(x). \]

**DEFINITION 7.** Let \( \varepsilon \) be an idempotent in \( \mathcal{C}\hat{\mathcal{G}} \), then we say \( L \) is a good group containing \( \varepsilon \) if \( L \) is a locally compact (in the \( \Delta_G \)-topology) subgroup of \( H(\varepsilon) \) and \( L \supset \rho^\varepsilon \hat{G} \).

We will show that the good groups arise exactly as the dual groups of the \( G_H \)'s, the abstract group \( G \) furnished with LCA topologies so that \( \text{id}: G_H \to G \) is continuous. Further there is at most one good group containing an idempotent.

**PROPOSITION 8.** Let \( \varepsilon \) be an idempotent in \( \mathcal{C}\hat{\mathcal{G}} \) and let \( L \) be a good group containing \( \varepsilon \), then \( \hat{\rho}_\varepsilon: \hat{L} \to \hat{G} \) is a monomorphism onto \( G \).

Proof. Since \( \rho^\varepsilon : \hat{G} \to L \) has dense range, we see that \( \hat{\rho}_\varepsilon \) is a monomorphism. Consider the map \( j \) defined in Proposition 7 restricted to \( L \). Then \( j \circ \rho^\varepsilon = i \), where \( i: \hat{G} \to \mathcal{B}\hat{G} \). Passing to the duals, we obtain \( \hat{i} = \hat{\rho}_\varepsilon \circ \hat{j} \), and \( \hat{i}: \hat{G}_d \to G \). Hence \( \hat{\rho}_\varepsilon \) is onto. \( \square \)

For notational convenience put \( \phi = \hat{\rho}_\varepsilon^* \). Applying Proposition 1 to the above situation we obtain that \( \phi \) is an isometric isomorphism of \( M(\hat{L}) \) onto \( M(G^L_\varepsilon) \). We now show that the Raikov projection induced by \( \hat{\rho}_\varepsilon \) is actually \( E_\varepsilon \).

**THEOREM 9.** Let \( \varepsilon \) be an idempotent in \( \mathcal{C}\hat{\mathcal{G}} \), and let \( L \) be a good group containing \( \varepsilon \). Then \( E_\varepsilon \) is the Raikov projection induced by \( \hat{\rho}_\varepsilon \), and \( \hat{\rho}_\varepsilon \) is isomorphic under \( \hat{\rho}_\varepsilon \) to \( G^L_\varepsilon \), \( G \) with a stronger
LCA topology.

Proof. The action of \( \phi \) can be described as follows: for \( \mu \in M(\hat{L}), \gamma \in \hat{G} \), \( (\phi \mu)(\gamma) = \hat{\mu}(\gamma \times \varepsilon) \). First we show that \( \phi M(\hat{L}) \subseteq E_\varepsilon M(G) \). Since \( \varepsilon \in \text{c} \hat{G} \), there exists a net \( \{ \gamma_\alpha \} \subseteq \hat{G} \) such that \( \gamma_\alpha \to \varepsilon \). Let \( \mu \in M(\hat{L}), \gamma \in \hat{G} \) then

\[
(\phi \mu)(\gamma_\alpha) = \hat{\mu}(\gamma_\alpha \times \gamma \times \varepsilon) = \hat{\mu}(\varepsilon \times \gamma \times \varepsilon) = (\phi \mu)(\gamma)
\]

since \( \hat{\mu} \) is continuous on \( L \). But

\[
(\phi \mu)(\varepsilon \times \gamma) = (\phi \mu)(\varepsilon \times \gamma)
\]

hence \( (\phi \mu)(\varepsilon \times \gamma) = (\phi \mu)(\gamma) \), and so \( E_\varepsilon \phi \mu = \phi \mu \). To show that

\[
E_\varepsilon M(G) \subseteq \phi M(\hat{L}), \text{ let } \mu \text{ be a positive measure in } E_\varepsilon M(G).
\]

To see that these positive measures span

\[
E_\varepsilon M(G), \text{ observe that } E_\varepsilon \mu = \int_\varepsilon^\lambda \mu, \text{ for } \mu \in M(G),
\]

\( \lambda = \exp |\mu|, \text{ and } \int_\varepsilon^\lambda > 0; \text{ hence } \mu > 0 \) implies

\[
E_\varepsilon \mu > 0.
\]

Then the function \( \hat{\mu}|L \) is positive definite on the dense subgroup \( \rho_\varepsilon \hat{G} \) of \( L \) and is continuous and bounded on \( L \); note for \( \gamma \in \hat{G}, \hat{\mu}(\gamma \times \varepsilon) = (E_\varepsilon \mu)(\gamma) = \hat{\mu}(\gamma) \). Hence \( \hat{\mu}|L \) is positive definite on \( L \), (suppose \( f \) is a continuous function on a topological group \( X \) and is positive definite on a dense subgroup \( Y \); then for \( n=1,2,..., c_i \in \mathbb{C} \), \( x_i \in X \) for \( i=1,...,n \) the expression

\[
\sum_{i,j} c_i c_j \int f(x_i^{-1}x_j)
\]

can be approximated by expressions of the form

\[
\sum_{i,j} c_i c_j \int f(y_j^{-1}y_i), \text{ for each } i=1,...,n \text{ respectively,}
\]

thus \( f \) is positive definite on \( X \), and by Bochner's theorem (see [7, p. 19]) there exists

\[
\nu \in M(\hat{L}) \text{ such that } \hat{\nu}(\varepsilon \times \gamma) = \hat{\nu}(\varepsilon \times \gamma) \text{ for each } \gamma \in \hat{G} .
\]

Clearly \( \mu = \phi \nu \).
Let $P$ be the Raikov projection induced by $\hat{\rho}_\varepsilon$, but then $P \ M(G) = M(G^\wedge_L) = \phi M(L) = E_\varepsilon M(G)$. Since $P$ and $E_\varepsilon$ commute, this shows that $P = E_\varepsilon$. □

Conversely, let $G_H$ be the abstract group $G$ with an LCA topology stronger than that of $G$. Let $P = E_\varepsilon$ be the Raikov projection induced by id: $G_H \rightarrow G$, then $\varepsilon$ is an idempotent in $c\ell \hat{G}$. (see Theorem 4). Now consider $\hat{G}$ as a subgroup of $\hat{G}_H$ (both groups are groups of characters of the abstract group $G$), then the morphism $\rho_\varepsilon : \hat{G} \rightarrow H(\varepsilon)$ extends to $\hat{G}_H$ in the following way: let $\chi \in \hat{G}_H$, then put $\rho_\varepsilon \chi = \sigma$, where $\sigma$ is defined by $f^\mu_\sigma = \chi f^\mu_\varepsilon$, for $\mu = \exp |\nu|$, $\nu \in M(G)$.

**THEOREM 10.** Under the preceding hypotheses, $\rho_\varepsilon \hat{G}_H$ is a good group containing $\varepsilon$. Further $\rho_\varepsilon \hat{G}_H = H(\varepsilon)$, and $H(\varepsilon)$ is the maximal subgroup of $\Delta_G$ containing $\varepsilon$.

Proof. The topology on $\hat{G}_H$ is induced by the Fourier-Stieltjes transforms of the elements of $M(G_H)$. The $\Delta_G$-topology on $\rho_\varepsilon \hat{G}_H$ is induced by the functions $\hat{\mu}, \mu \in M(G)$. But for $\mu \in M(G), \chi \in \hat{G}_H$, we have $\hat{\mu}(\rho_\varepsilon \chi) = \hat{\mu}(\varepsilon \times \rho_\varepsilon \chi) = (E_\varepsilon \mu)^\wedge(\rho_\varepsilon \chi) = (E_\varepsilon \mu)^\wedge(\chi)$, where the last term is the $G_H$-Fourier-Stieltjes transform of $E_\varepsilon \mu \in M(G_H)$. Hence $\rho_\varepsilon \hat{G}_H$ in the $\Delta_G$-topology is homeomorphic to $\hat{G}_H$, so $\rho_\varepsilon \hat{G}_H$ is LCA, and contains $\rho_\varepsilon \hat{G}$. Thus $\rho_\varepsilon \hat{G}_H$ is a good group.

The topologies on the maximal ideal space $\Delta_{G_H}$ of $M(G_H)$ and on $\varepsilon \times \Delta_G$ are both induced by the set $\{\hat{\mu}: \mu \in M(G_H) = E_\varepsilon M(G)\}$. It follows that $\Delta_{G_H}$
is homeomorphic to $\varepsilon \times \Delta_G$. Indeed, a generalized character argument shows that $\Delta_{G_H}$ is isomorphic to $\varepsilon \times \Delta_G$ as a semitopological semigroup (see the remark immediately preceding Theorem 10).

The annihilator of $L^1(G_H)$ in $\varepsilon \times \Delta_G$ is $(\varepsilon \times \Delta_G) \setminus \rho \hat{G}_H$. Now let $\sigma \in \Delta_G$ with $\sigma \times \varepsilon = \sigma$ (that is, $\sigma \in \varepsilon \times \Delta_G$), and suppose there exists $\tau \in \Delta_G$ such that $\sigma \times \tau = \varepsilon$. We now show that $\sigma \in \rho \hat{G}_H$. For $\mu = \exp|\lambda|$, $\lambda \in M(G)$, we have $|f_{\sigma}^\mu| = f_{\varepsilon}^\mu$. In particular, let $\lambda \in L^1(G_H)$, $\lambda \not= 0$, and $\mu = \exp|\lambda|$. Then $f_{\varepsilon}^\mu = 1$ ($|\mu|$ almost everywhere), so $|f_{\sigma}^\mu| = 1$ ($|\mu|$ almost everywhere). Thus $f_{\sigma}^\mu \in L^1(\lambda)$, and there exists a bounded continuous function $g$ on $G_H$ such that $\int_{G_H} f_{\sigma}^\mu gd\lambda \neq 0$. But $g_{\lambda} \in L^1(G_H)$ and so $\sigma \in \rho \hat{G}_H$. $\square$

**Corollary 11.** Let $\varepsilon$ be an idempotent in $\mathcal{G}$, and let $L$ be a good group containing $\varepsilon$. Then $L = H(\varepsilon)$, and $H(\varepsilon)$ is the maximal subgroup of $\Delta_G$ containing $\varepsilon$. Hence $H(\varepsilon)$ is the unique good group containing $\varepsilon$.

**Proof.** Apply Theorem 10 to $G_L$. $\square$

Taylor [9] showed that the maximal subgroup of $\Delta_G$ containing $1$ is $\hat{G}$. In a future paper the authors will present an example of an LCA group $G$ such that $\mathcal{G}$ contains an idempotent $\varepsilon$ with $H(\varepsilon)$ not locally compact.
REFERENCES


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