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MULTIPLIERS ON COMPACT GROUPS[1]

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ABSTRACT. Let a compact group G act continuously both by left and right translation on a Banach space V of integrable functions on G. Then $\mathfrak{M}(V)$, the space of bounded linear operators on V commuting with right translation, contains a homomorphic image of $L^1(G)$, whose closure is exactly the set of operators on which G acts continuously. Further, this set is exactly the ideal of compact operators in $\mathfrak{M}(V)$. A restricted version holds for noncompact groups.

1. **Compact groups.** In this section G denotes a compact group with normalized Haar measure m, and the space $L^p(G, m)$, $1 \leqq p < \infty$, is briefly denoted by $L^p(G)$. We denote the algebra of finite regular Borel measures on G by $M(G)$.

Let V be a Banach space of functions contained in $L^1(G)$ which is closed under left and right translations.

DEFINITION. We say that V is a G-G module if for each $x \in G$, $L(x)f \in V$ and $R(x)f \in V$, and $\|L(x)f-f\|_V \to 0$ and $\|R(x)f-f\|_V \to 0$ as $x \to e$ for each $f \in V$ (the translations $L(x)$ and $R(x)$ are given by $L(x)f(y) = f(x^{-1}y)$, $R(x)f(y) = f(yx)$, $x, y \in G, f \in V$). Furthermore, we require that $\|L(x)f\|_V = \|f\|_V$ and $\|R(x)f\|_V = \|f\|_V$ for each $x \in G, f \in V$.

Henceforth V will be a G-G module.

As Rieffel [2, p. 447] points out, V is also an $M(G)$-$M(G)$ module, that is, V is closed under left and right convolution by measures.

Now let $\hat{G}$ be the dual of G, namely, the set of equivalence classes of continuous unitary irreducible representations of G. For $\alpha \in \hat{G}$, let T_α be an element of α. Then T_α is a continuous homomorphism of G into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices. Let $\chi_\alpha(x) = \operatorname{Trace}(T_\alpha(x))$, the character of α, and let W_α be the linear span of the matrix entry functions of T_α. Then χ_α and W_α depend only on α. We call an element in the linear span of $\{W_\alpha : \alpha \in \hat{G}\}$ a trig polynomial.

We note here for later use that $L^1(G)$ has a bounded approximate identity $\{t_\iota\}$ which is central, that is, $t_\iota * f = f * t_\iota$, $f \in L^1(G)$. This follows since G has a base of invariant neighborhoods of the identity.

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Thus $L^1(G)$ has a bounded central approximate identity consisting of trig polynomials.

For $\alpha \in \hat{G}$, $f \in V$, we have that $f * \chi_\alpha \in W_\alpha \cap V$. Since V is left and right invariant, it further holds that $W_\alpha \cap V = W_\alpha$ or $\{0\}$.

DEFINITION. Let $\mathfrak{M}(V)$ be the space of bounded operators on V which commute with all right translations. Denote the operator norm by $\|\cdot\|_{\mathrm{op}}$. For $\mu \in M(G)$, define the operator $j(\mu)$ on V by $j(\mu)f = \mu * f$, $f \in V$.

Note that for $T \in \mathfrak{M}(V)$, $\mu \in M(G)$, $f \in V$ that $T(f * \mu) = (Tf) * \mu$.

COROLLARY 1. *The map j is a bounded homomorphism of $M(G)$ into $\mathfrak{M}(V)$.*

Now $\mathfrak{M}(V)$ is a right $L^1(G)$-module, and the action is given by $(T \cdot g)(f) = T(g * f)$, for $T \in \mathfrak{M}(V)$, $g \in L^1(G)$, $f \in V$. That is, $T \cdot g$ is nothing but $Tj(g)$ (operator composition).

DEFINITION (RIEFFEL [2, p. 454]). The essential part of $\mathfrak{M}(V)$, denoted by $\mathfrak{M}_e(V)$, is the closed span of $\{T \cdot f : T \in \mathfrak{M}(V), f \in L^1(G)\}$. That is, $\mathfrak{M}_e(V)$ is just the closed left ideal generated by $jL^1(G)$.

THEOREM 2 (COHEN, RIEFFEL [2, p. 454]). *The space*

$$\mathfrak{M}_e(V) = \mathfrak{M}(V)L^1(G).$$

For $x \in G$, let δ_x be the unit mass at x; then for $f \in V$, $\delta_x * f = L(x)f$. Now G acts in $\mathfrak{M}(V)$ by $T \mapsto Tj(\delta_x)$ for $T \in \mathfrak{M}(V)$. Our aim is to characterize those $T \in \mathfrak{M}(V)$ for which $\|Tj(\delta_x) - T\|_{\mathrm{op}} \to 0$ as $x \to e$. As Rieffel [2, p. 456] observes, these operators are exactly those in the essential part of $\mathfrak{M}(V)$.

LEMMA 3. *Let g be a trig polynomial on G and let $T \in \mathfrak{M}(V)$. Then $T \cdot g = Tj(g) = j(k)$ for some trig polynomial k.*

PROOF. Let E be a finite set contained in $\hat{G}$ which carries g, that is $g = \sum_{\alpha \in E} n_\alpha g * \chi_\alpha$. Thus

$$\begin{aligned} T \cdot g(f) &= T\left(g * \left(\sum_{\alpha \in E} n_\alpha \chi_\alpha * f\right)\right) \\ &= T\left(g * \left(f * \sum_{\alpha \in E} n_\alpha \chi_\alpha\right)\right) \\ &= (T \cdot g(f)) * \sum_{\alpha \in E} n_\alpha \chi_\alpha \end{aligned}$$

which is in V_E, the span of $\{V \cap W_\alpha : \alpha \in E\}$. Now V_E is a finite dimensional G-G module, and $T\cdot g$ is an operator on V_E which commutes with right translation. Thus there exists a trig polynomial h such that $T\cdot g(f)=h*f$ for all $f\in V_E$. But for any $f\in V$,

$$T\cdot g(f) = T\cdot g\left(\sum_{\alpha\in E} n_\alpha\chi_\alpha * f\right) = j(h)\left(\sum_{\alpha\in E} n_\alpha\chi_\alpha * f\right)$$

$$= j\left(h * \sum_{\alpha\in E} n_\alpha\chi_\alpha\right)(f). \quad \square$$

THEOREM 4. *With hypotheses and notation as stated above,* $\mathfrak{M}_e(V) = \text{closure}(jL^1(G))$.

PROOF. If $g\in L^1(G)$, then by the Cohen factorization theorem $g=g_1 * g_2$, $g_1, g_2\in L^1(G)$. Thus $j(g)=j(g_1)j(g_2)=j(g_1)\cdot g_2\in\mathfrak{M}(V)\cdot L^1(G)$. (Alternatively, in a not so high-powered fashion, observe directly that $\|j(g)j(\delta_x)-j(g)\|_{\text{op}}\leqq\|g*\delta_x-g\|_1=\|R(x^{-1})g-g\|_1\to 0$ as $x\to e$.) Thus $jL^1(G)\subset\mathfrak{M}_e(V)$, a closed set.

Conversely, let $T\in\mathfrak{M}_e(V)$, then $T=S\cdot f$ for some $S\in\mathfrak{M}(V)$, $f\in L^1(G)$. Let $\{t_\iota\}$ be the bounded central approximate identity consisting of trig polynomials mentioned above. Then

$$\|T - T\cdot t_\iota\|_{\text{op}} = \|Sj(f) - Sj(f * t_\iota)\|_{\text{op}}$$

$$\leqq \|S\|_{\text{op}}\|f - f * t_\iota\|_1 \xrightarrow{\iota} 0.$$

By the lemma, $T\cdot t_\iota\in jL^1(G)$. $\square$

THEOREM 5. *The ideal of compact operators in* $\mathfrak{M}(V)$ *is equal to* $\mathfrak{M}_e(V)$.

PROOF. By the above, $\mathfrak{M}_e(V)=\text{closure}(jL^1(G))$. If $f\in L^1(G)$ then $\|j(f)-j(f*t_\iota)\|_{\text{op}}\leqq\|f-f*t_\iota\|_1\xrightarrow{\iota}0$. Each $j(f*t_\iota)$ is an operator of finite rank, thus $j(f)$ is compact. The fact that the set of compact operators is norm closed gives containment one way.

Recall the fact that if $\{P_\iota\}$ is a norm-bounded net of bounded operators on a Banach space X converging strongly to the identity (that is, $P_\iota x\xrightarrow{\iota}x$, each $x\in X$) and if T is a compact operator on X, then $\|P_\iota T-T\|_{\text{op}}\xrightarrow{\iota}0$.

Let h be a central trig polynomial, $T\in\mathfrak{M}(V)$; then $j(h)T=Tj(h)$. In fact, if $f\in V$, then $(j(h)T)(f)=h*(Tf)=(Tf)*h=T(f*h)=T(h*f)$. Now let T be a compact operator in $\mathfrak{M}(V)$. We will show that $\|T-T\cdot t_\iota\|_{\text{op}}\to 0$ and thus $T\in\mathfrak{M}_e(V)$.

Let $f\in V$, then by the Cohen factorization theorem there exist $g\in L^1(G)$, $f_1\in V$ such that $f=g*f_1$. Now

$$\|j(t_\iota)f - f\|_V = \|j(t_\iota * g)(f_1) - j(g)(f_1)\|_V$$

$$\leqq \|t_\iota * g - g\|_1 \|f_1\|_V \xrightarrow{\iota} 0,$$

thus $\{j(t_\iota)\}$ converges strongly to the identity in $\mathfrak{M}(V)$. So $\|T - T\cdot t_\iota\|_{\mathrm{op}} = \|T - Tj(t_\iota)\|_{\mathrm{op}} = \|T - j(t_\iota)T\|_{\mathrm{op}} \to 0$. □

COROLLARY 6. *For $T \in \mathfrak{M}(V)$ the following are equivalent:*

(1) $\|Tj(\delta_x) - T\|_{\mathrm{op}} \to 0$ *as* $x \to e$,

(2) $T = S\cdot g$, *some* $S \in \mathfrak{M}(V)$, $g \in L^1(G)$,

(3) $T \in \mathrm{closure}(jL^1(G))$,

(4) *T is a compact operator.*

APPLICATIONS. Let $1 < p < \infty$, and $V = L^p(G)$; then $\mathfrak{M}(V)$ is the multiplier algebra of $L^p(G)$. As a particular example, consider $V = L^2(T)$ (T is the circle group); then $\mathfrak{M}(V)$ is identified with $l^\infty(Z)$, and $\mathfrak{M}_e(V)$ consists of those bounded sequences $\{\phi_n\}$ for which $\sup_n |\phi_n - \phi_n e^{-inx}| \to 0$ as $x \to 0$, namely $c_0(Z)$, the sup-norm closure of $L^1(T)\hat{}$. For a compact group G and $V = L^2(G)$ we get $\mathfrak{M}(V) = \mathfrak{L}^\infty(\hat{G})$ (see [1]), and $\mathfrak{M}_e(V) = \mathfrak{C}_0(\hat{G})$. For $V = C(G)$, $\mathfrak{M}(V) = M(G)$ and $\mathfrak{M}_e(V) = L^1(G)$.

2. **Locally compact groups.** Here G will be a noncompact locally compact group, $L^1(G)$ the ideal of finite regular Borel measures absolutely continuous with respect to left invariant Haar measure. Theorem 4 does not hold in general in this context. For example, for the real line R, consider $\mathfrak{M}(L^2(\hat{R})) = L^\infty(R)$, then the essential part is $L_0^\infty(R) = L^\infty(R)\cdot C_0(R)$ which is strictly larger than $C_0(R) = L^1(\hat{R})\hat{}^{-}$. However it is true that $jM(G) \cap \mathfrak{M}_e(V) \subset \mathrm{closure}\, jL^1(G)$.

We will not require that V be a space of functions. Here V will be an isometric left G module with the action denoted xf ($x \in G, f \in V$), and $\mathfrak{M}(V)$ will denote the space of bounded operators on V. The map $j: M(G) \to \mathfrak{M}(V)$, given by $j(\mu)(f) = \int_G (xf)\, d\mu(x)$, $f \in V$, $\mu \in M(G)$, is a homomorphism with $\|j(\mu)\|_{\mathrm{op}} \leqq \|\mu\|$. The essential part of $\mathfrak{M}(V)$, denoted by $\mathfrak{M}_e(V)$, equals $\mathfrak{M}(V)(jL^1(G))$.

The following holds for $T \in \mathfrak{M}(V): T \in \mathfrak{M}_e(V)$ if and only if $\|Tj(\delta_x) - T\|_{\mathrm{op}} \to 0$ as $x \to e$.

THEOREM 7. $jM(G) \cap \mathfrak{M}_e(V) = jM(G) \cap \mathrm{closure}(jL^1(G))$.

PROOF. As before it is clear that $\mathrm{closure}(jL^1(G)) \subset \mathfrak{M}_e(V)$. Now let $\mu \in M(G)$ such that $j(\mu) \in \mathfrak{M}_e(V)$; then there exist $T \in \mathfrak{M}(V)$, $g \in L^1(G)$ such that $j(\mu) = T\cdot g$. Let $\{u_\iota\}$ be an approximate identity in $L^1(G)$, then $\mu * u_\iota \in L^1(G)$ for each ι and

$$\begin{aligned}\|j(\mu) - j(\mu * u_\iota)\|_{\mathrm{op}} &= \|T\cdot g - (T\cdot g)(j(u_\iota))\|_{\mathrm{op}} \\ &= \|Tj(g) - Tj(g)j(u_\iota)\|_{\mathrm{op}} \\ &\leqq \|T\|_{\mathrm{op}}\|g - g * u_\iota\|_1 \xrightarrow{\iota} 0. \quad \square\end{aligned}$$

COROLLARY 8. *Let* $\mu \in M(G)$, *then the following are equivalent*:
(1) $\|j(\mu * \delta_x) - j(\mu)\|_{\mathrm{op}} \to 0$ *as* $x \to e$,
(2) $j(\mu) \in \mathrm{closure}(jL^1(G))$.

APPLICATION. For $1 < p < \infty$, let $L^p(G)$ be the L^p space of left invariant Haar measure. Corollary 8 characterizes the measures which can be approximated in the L^p-operator norm by $L^1(G)$. Let V be the direct sum of all (classes of) irreducible unitary continuous representations of G; then the V-operator norm is the C^* norm $\|\cdot\|_{\hat{G}}$ of $L^1(G)$ and $M(G)$. Thus we have another proof of our characterization of $M_0(G)$, the measures approximable in $\|\cdot\|_{\hat{G}}$ by $L^1(G)$ (see [1]).

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