TRANSLATION IN MEASURE ALGEBRAS AND THE CORRESPONDENCE TO FOURIER TRANSFORMS VANISHING AT INFINITY

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Let $G$ denote a locally compact (not necessarily abelian) group and $M(G)$ the collection of finite regular Borel measures on $G$. The set $M(G)$ is a semisimple Banach algebra with identity under convolution *. It can be identified with the dual space of $C_0(G)$, the space of continuous complex-valued functions on $G$ that vanish at infinity, with the sup-norm. The group $G$ has a left-invariant regular Borel measure $dm(x)$ that is unique up to a constant and is called the left Haar measure of $G$. Let $C_b(G)$ denote the space of bounded continuous functions on $G$. For each $x \in G$, we define on $C_b(G)$ the left-translation operator by the relation

$$L(x)f(y) = f(x^{-1}y) \quad (f \in C_b(G)).$$

We say that $f \in C_b(G)$ is right uniformly continuous if $L(x_\alpha)f \to L(x)f$ uniformly, whenever $x_\alpha \to x$. Let $C_b^r(G)$ denote the subspace of $C_b(G)$ of right uniformly continuous functions. For $\mu \in M(G)$, define $L(x)\mu \in M(G)$ by the condition

$$\int_G f(t) dL(x)\mu(t) = \int_G L(x^{-1})f(t) d\mu(t),$$

where $f \in C_0(G)$. We wish to study for which $\mu \in M(G)$ the map $x \mapsto L(x)\mu$ is continuous from $G$ into $M(G)$, where $M(G)$ will be equipped with an $L(x)$-invariant metric topology. In particular, we shall characterize $M_0(G)$, the algebra of measures whose Fourier transform vanishes at infinity.

Let $A \subset C_b^r(G)$ be a linear subspace with sufficiently many elements to separate the points of $M(G)$; in other words, if $\mu \in M(G)$ and if

$$\int_G f(t) d\mu(t) = 0$$

for all $f \in A$, then $\mu = 0$. We are then able to pair $A$ and $M(G)$ by the relation

$$\langle f, \mu \rangle = \int_G f(t) d\mu(t) \quad (f \in A; \ \mu \in M(G)).$$

Let $\sigma(A, M(G))$ denote the weak topology on $A$ induced by this pairing. Suppose $A$ can be written as $\bigcup_{k=1}^\infty A_k$, where each $A_k$ is a subset of $A$ that is $L(x)$-invariant for all $x \in G$ and where each $A_k$ is $\sigma(A, M(G))$-bounded. Note that $A_k$ is

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\( \sigma(\mathcal{A}, \mathcal{M}(G)) \)-bounded if and only if \( A_k \) is bounded in sup-norm. We let \( \mathcal{T}(A_k) \) denote the topology on \( \mathcal{M}(G) \) of uniform convergence on the sets \( A_k \). Note that \( \mathcal{T}(A_k) \) gives an \( L(\mathcal{X}) \)-invariant metric topology on \( \mathcal{M}(G) \). For \( k \geq 1 \), let

\[
\tau_k(\mu) = \sup \left\{ \left| \left< f, \mu \right> \right| : f \in A_k \right\}.
\]

Then \( \tau_k \) is an \( L(\mathcal{X}) \)-invariant seminorm on \( \mathcal{M}(G) \).

**Definition.** For \( \mu \in \mathcal{M}(G) \), we say that \( \mu \) has separable orbit in \( (\mathcal{M}(G), \mathcal{T}(A_k)) \) if there exists a sequence \( \{x_n\}_{n=1}^{\infty} \subset \mathcal{G} \) such that for each \( x \in \mathcal{G}, k \geq 1, \) and \( \varepsilon > 0 \), there exists an \( x_n \) such that \( \tau_k(\mathcal{L}(x)\mu - \mathcal{L}(x_n)\mu) < \varepsilon \).

**PROPOSITION 1.** Let \( \mu \in \mathcal{M}(G) \) have separable orbit in \( (\mathcal{M}(G), \mathcal{T}(A_k)) \). Then \( s \mapsto \mathcal{L}(s)\mu \) is continuous from \( \mathcal{G} \) to \( (\mathcal{M}(G), \mathcal{T}(A_k)) \).

**Proof.** Let \( s_\alpha \xrightarrow{\alpha} s \). Choose \( k \geq 1 \) and \( \varepsilon > 0 \). We need to show that there exists \( \alpha_0 \) such that for \( \alpha \geq \alpha_0 \), we have the inequality \( \tau_k(\mathcal{L}(s_\alpha)\mu - \mathcal{L}(s)\mu) < \varepsilon \). Note that for \( f \in C_{\mathcal{T}}(\mathcal{G}), y \beta, y \beta^{-1} \), \( \mathcal{L}(y)\mu \) uniformly as \( y \beta, y \beta^{-1} \) (and hence as \( y \beta^{-1} \)). Thus

\[
\left< f, \mathcal{L}(y)\mu \right> = \left< \mathcal{L}(y^{-1})f, \mu \right> + \left< \mathcal{L}(y^{-1})f, \mu \right> = \left< f, \mathcal{L}(y)\mu \right>.
\]

Let \( S(n) = \{ y \in \mathcal{G} : \tau_k(\mathcal{L}(y)\mu - \mathcal{L}(x_n)\mu) \leq \varepsilon/3 \} \). We wish to show that \( S(n) \) is closed. Let \( y_\beta \in S(n) \) be such that \( y_\beta \beta, y \). Thus

\[
\tau_k(\mathcal{L}(y)\mu - \mathcal{L}(x_n)\mu) = \sup \left\{ \lim_{\beta} \left| \left< f, \mathcal{L}(y)\mu - \mathcal{L}(x_n)\mu \right> \right| : f \in A_k \right\} \leq \varepsilon/3.
\]

Hence \( S(n) \) is closed.

By hypothesis, \( \mathcal{G} = \bigcup_{n=1}^{\infty} S(n) \). By the Baire category theorem for locally compact groups, there exists \( n_0 \) such that \( S(n_0) \) has an interior. Thus there exists an open set \( U \) about \( s \) such that \( t_0 s^{-1} U \subset S(n_0) \) for some \( t_0 \in S(n_0) \). Let \( \alpha_0 \) be such that \( s_\alpha \in U \) for \( \alpha \geq \alpha_0 \). We now show that for \( \alpha \geq \alpha_0 \), the inequality

\[
\tau_k(\mathcal{L}(s_\alpha)\mu - \mathcal{L}(s)\mu) < \varepsilon
\]

holds. For \( \alpha \geq \alpha_0 \), we have that

\[
\tau_k(\mathcal{L}(s_\alpha)\mu - \mathcal{L}(s)\mu) = \tau_k(\mathcal{L}(t_0 s^{-1})\mathcal{L}(s_\alpha)\mu - \mathcal{L}(t_0 s^{-1})\mathcal{L}(s)\mu)
\]

\[
\leq \tau_k(\mathcal{L}(t_0 s^{-1} s_\alpha)\mu - \mathcal{L}(x_n)\mu) + \tau_k(\mathcal{L}(x_n)\mu - \mathcal{L}(t_0)\mu)
\]

\[
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,
\]

since \( t_0, t_0 s^{-1} s_\alpha \in t_0 s^{-1} U \subset S(n_0) \).

**PROPOSITION 2.** Let \( G \) be \( \sigma \)-compact. If \( x \mapsto \mathcal{L}(x)\mu \) is continuous from \( \mathcal{G} \) to \( (\mathcal{M}(G), \mathcal{T}(A_k)) \), then \( \mu \) has separable orbit in \( (\mathcal{M}(G), \mathcal{T}(A_k)) \).

**Proof.** Note that \( (\mathcal{M}(G), \mathcal{T}(A_k)) \) is a metric space. Let \( G = \bigcup_{n=1}^{\infty} K_n \), where \( K_n \) is compact. The image of \( K_n \) under \( x \mapsto \mathcal{L}(x)\mu \) is a compact metric space and hence is separable. Thus the image of \( G \) is separable.
If \( G \) is not \( \sigma \)-compact and \( M(G) \) has the measure norm topology, then no non-zero measure has a separable orbit.

We now show that \( \mu \in M(G) \) has the property that \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \mathcal{F}(A_k)) \) if and only if \( \mu \) is in the \( \mathcal{F}(A_k) \)-closure of \( L^1(G) \), denoted by \( L^1(\overline{G})^A \).

**Theorem 3.** Let \( \mu \in M(G) \) be such that \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \mathcal{F}(A_k)) \). Then \( \mu \in L^1(\overline{G})^A \).

**Proof.** Let \( \{f_\alpha\} \) be an approximate identity in \( L^1(G) \), indexed over a neighborhood of \( e \); in other words, support \( \{f_\alpha\} \subset \alpha \), \( f_\alpha \geq 0 \), and \( \|f_\alpha\|_1 = 1 \). Choose \( k_0 \geq 1 \) and \( \varepsilon > 0 \). It suffices to show that \( \tau_k(f_\alpha * \mu - \mu) \leq \varepsilon \) for \( \alpha \geq \alpha_0 \), for some \( \alpha_0 \). Pick \( U \) to be a symmetric neighborhood of \( e \) in \( G \) such that

\[
\tau_k(L(x)\mu - \mu) < \varepsilon
\]

for \( x \in U \). Choose \( \alpha_0 \) such that the inequality \( \alpha \geq \alpha_0 \) implies that support \( \{f_\alpha\} \subset U \). Now for \( \alpha \geq \alpha_0 \),

\[
\tau_k(f_\alpha * \mu - \mu) = \sup \left\{ \left| \left< \phi, f_\alpha * \mu - \mu \right> \right| : \phi \in A_k \right\}
= \sup \left\{ \left| \int \phi(x) f_\alpha(x) \mu(x) - \int \phi(y) d\mu(y) \right| : \phi \in A_k \right\}
= \sup \left\{ \left| \int \int \phi(x) d\mu(x)f_\alpha(x) - \int \phi(y) d\mu(y) \right| : \phi \in A_k \right\}
= \sup \left\{ \left| \int \phi(y) dL(x)\mu(y) - \int \phi(y) d\mu(y) \right| : \phi \in A_k \right\}
\leq \sup_{x \in U} \tau_k(L(x)\mu - \mu) \leq \varepsilon.
\]

**Theorem 4.** Let \( \mu \in L^1(\overline{G})^A \). Then \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \mathcal{F}(A_k)) \).

**Proof.** We note first that since \( A_k \) is \( \sigma(A, M(G)) \)-bounded and \( L(x) \)-invariant, \( A_k \) is a sup-norm bounded set in \( C^B(G) \); in fact, for all \( x \in G \), we have that

\[
\sup_{f \in A_k} \left| f(x) \right| = \sup_{f \in A_k} \left| \int L(x^{-1}) f(x) d\delta_e \right| = \sup_{f \in A_k} \left| \int f(x) d\delta_e \right| = M < \infty,
\]

where \( \delta_e \) is the unit mass at \( e \). Now \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( M(G) \) in the measure norm, for \( \mu \in L^1(G) \). Thus, since \( A_k \) is a sup-norm bounded set, \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \mathcal{F}(A_k)) \) for \( \mu \in L^1(G) \). Choose \( \mu \in L^1(\overline{G})^A \), and let \( x_{\alpha} \overset{\alpha}{\to} x \). Let \( k \geq 1 \) and \( \varepsilon > 0 \). We need to find an \( \alpha_0 \) such that if \( \alpha_0 \leq \alpha \), then \( \tau_k(L(x_{\alpha})\mu - L(x)\mu) < \varepsilon \). First pick \( f \in L^1(G) \) such that \( \tau_k(f - \mu) < \varepsilon/3 \). Now choose \( \alpha_0 \) such that for \( \alpha \geq \alpha_0 \), \( \tau_k(L(x_{\alpha})f - L(x)f) < \varepsilon/3 \). Thus for \( \alpha \geq \alpha_0 \),
\[ \tau_k(L(x,\alpha)\mu - L(x)\mu) \leq \tau_k(L(x,\alpha)\mu - L(x)\mu) + \tau_k(L(x,\alpha)f - L(x)f) + \tau_k(L(x)f - L(x)\mu) < \tau_k(\mu - f) + \frac{\varepsilon}{3} + \tau_k(f - \mu) < \varepsilon. \]

Remark. The two theorems above also hold if \( A \) is a space of bounded Borel functions, rather than a subspace of \( C^0_r(G) \).

For \( \mu \in M(G) \), let \( \| \mu \| \) denote the measure norm of \( \mu \), that is, the norm of \( \mu \) as a linear functional on \( C_0(G) \) with sup-norm \( \| f \|_\infty = \sup \{ |f(x)| : x \in G \} \). If we let \( A_k = \{ f \in C_0(G) : \| f \|_\infty < k \} \), then \( \mathcal{F}(A_k) \) is the measure norm topology. Thus we have the following corollaries.

**Corollary 5.** Let \( \mu \in M(G) \). If \( \mu \) has separable orbit in \( (M(G), \| \cdot \|) \), then \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \| \cdot \|) \).

Suppose \( G \) is \( \sigma \)-compact. If \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \| \cdot \|) \), then \( \mu \) has separable orbit in \( (M(G), \| \cdot \|) \).

**Corollary 6.** Let \( \mu \in M(G) \). The measure \( \mu \) is absolutely continuous if and only if \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \| \cdot \|) \).

Remarks. Propositions 1 and 2 are similar in spirit to a theorem of K. Shiga [8] in the compact case. Corollary 5 was obtained by R. Larsen [5] for the case where \( G \) is second countable and by K. W. Tam [9] in the general case. Corollary 6 was obtained by W. Rudin [7].

We now study \( M(G) \) under its sup-norm \( \| \cdot \|_\infty \). We shall give first the abelian case for motivation. We then treat the compact nonabelian case and finally the general case.

Let \( G \) be abelian, and let \( \hat{G} \) denote the character group of \( G \). For \( \mu \in M(G) \), define \( \hat{\mu}(\gamma) = \int_G \overline{\gamma(x)}d\mu(x) \), for \( \gamma \in \hat{G} \). Then \( \hat{\mu} \) is the Fourier transform of \( \mu \). For \( \mu \in M(G) \), let

\[ \| \mu \|_\infty = \sup \{ |\hat{\mu}(\gamma)| : \gamma \in \hat{G} \} \]

Let \( M_0(G) = \{ \mu \in M(G) : \mu \in C_0(\hat{G}) \} \).

**Corollary 7.** Let \( G \) be abelian. The map \( x \mapsto L(x)\mu \) is continuous from \( G \) to \( (M(G), \| \cdot \|_\infty) \) if and only if \( \mu \in M_0(G) \).

**Proof.** Let \( A_k = \{ f : f \in L^1(\hat{G}) \text{ with } \| f \|_1 < k \} \). Then \( \mathcal{F}(A_k) \) is the topology of \( (M(G), \| \cdot \|_\infty) \).

Remark. Corollary 7 was obtained by R. Goldberg and A. Simon [3]. They used the following result: If \( U \) is a relatively compact neighborhood of \( 0 \) in \( G \) (where \( G \) is abelian), there exists a compact subset \( K \) of \( \hat{G} \) such that for \( \gamma \in \hat{G} \setminus K \), there exists an \( x \in U \) with \( \gamma(x) \leq 0 \). To see this, let \( \sqrt{2} \leq \delta < \sqrt{3} \), and define \( U^0 = \{ \gamma \in \hat{G} : |\gamma(x) - 1| < \delta \text{ for all } x \in U \} \). Note that \( U^0 \) is relatively compact in \( \hat{G} \) (K. H. Hofmann and P. S. Mostert [4, p. 284] or Pontryagin [6, p. 237]). Let \( K \) be the closure of \( U^0 \) in \( \hat{G} \). We now prove the analogous result for the case where \( G \) is compact and nonabelian. This result is independent of the rest of this paper. We use the notation of Dunkl and Ramirez [1, Chapters 7 and 8], where proofs of unproved statements below may be found.
Let $G$ be a compact, nonabelian group. We let $\hat{G}$ denote the set of equivalence classes of continuous, unitary irreducible representations of $G$. For $\alpha \in \hat{G}$, let $T_\alpha$ be an element of $\alpha$. Then $T_\alpha$ is a homomorphism of $G$ into $U(n_\alpha)$, the group of unitary $n_\alpha \times n_\alpha$ matrices, where $n_\alpha$ is the dimension of $\alpha$. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$ ($1 \leq i, j \leq n$) and $T_{\alpha ij}$ to denote the function $x \mapsto T_\alpha(x)_{ij}$. Clearly
\[
T_\alpha(xy)_{ij} = \sum_{k=1}^{n_\alpha} T_\alpha(x)_{ik} T_\alpha(y)_{kj} \quad \text{and} \quad T_\alpha(y^{-1})_{ij} = T_\alpha(y)_{ji}.
\]
Furthermore, $T_{\alpha ij} \in C(G)$, where $C(G)$ denotes the set of continuous functions on $G$. For $\alpha \in \hat{G}$, let
\[
\chi_\alpha(x) = \text{trace } (T_\alpha(x)) = \sum_{i=1}^{n_\alpha} T_\alpha(x)_{ii}.
\]
This trace $\chi_\alpha$ is called the character of $\alpha$, and it is independent of the choice of $T_\alpha$ in $\alpha$. Let $X$ be an $n$-dimensional, complex inner-product space. Let $B(X)$ denote the space of linear maps from $X$ into $X$. We define the operator norm of $A \in B(X)$ by
\[
\|A\|_\infty = \sup \{ |A\xi| : \xi \in X, \|\xi\| \leq 1 \}.
\]
For the trace of $A$, we find that $\text{Tr } A = \sum_{i=1}^{n} (A\xi_i, \xi_i)$, where $\{\xi_i\}_{i=1}^{n}$ is some orthonormal basis for $X$ and $(\cdot, \cdot)$ denotes the inner product in $X$. Let $|A|$ denote $(A^*A)^{1/2}$. The operator norm of $A$ is $\|A\|_\infty$, that is, $\max \{\lambda_i : 1 \leq i \leq n\}$, where the $\lambda_i$ are the eigenvalues of $|A|$. For each $A \in B(X)$, we have the inequality $|\text{tr } A| \leq n\|A\|_\infty$.

PROPOSITION 8. Let $G$ be a compact group. Suppose $0 < \delta < \sqrt{3}$, and let $U$ be a neighborhood of $e$ in $G$. Let $U^0 = \{ \alpha \in \hat{G} : \|T_\alpha(x) - I\|_\infty < \delta \text{ for all } x \in U \}$. Then $U^0$ is finite.

Proof. We show that $U^0$ is an equicontinuous set of representations of $G$. Choose $\varepsilon > 0$. Let $K$ be a positive constant such that for $0 \leq \theta \leq 2\pi/\delta$, we have the inequality $|e^{i\theta} - 1| \leq K\delta$ (for example, let $K = 3\pi \sqrt{3}/2$). Define
\[
V_m = \{ x \in G : x, x^2, \ldots, x^m \in U \}.
\]
Clearly, $V_m$ is a neighborhood of $e$ in $G$. Pick $m$ such that $K\delta/m < \varepsilon$. Then for $x_1, x_2 \in G$ with $x = x_1^{-1}x_2 \in V_m$, we have that
\[
\|T_\alpha(x_1) - T_\alpha(x_2)\|_\infty = \|I - T_\alpha(x_1^{-1}x_2)\|_\infty = \|I - T_\alpha(x)\|_\infty
\]
\[
= \sup \{|1 - e^{i\theta}j| : 1 \leq j \leq n_\alpha\} \quad (\alpha \in U^0),
\]
by diagonalizing $T_\alpha(x)$. Thus
\[
\|I - T_\alpha(x^r)\|_\infty = \sup \{|1 - e^{ir\theta}j| : 1 \leq j \leq n_\alpha\} < \delta
\]
for $1 \leq r \leq m$. Therefore
\[ \| I - T_{\alpha}(x) \|_\infty = \sup \{ |1 - e^{i\theta_j}| : 1 \leq j \leq n_\alpha \} < K\delta / m < \varepsilon. \]

Thus \( U^0 \) is an equicontinuous set of representation of \( G \).

Let \( \chi_\alpha = \text{Tr} T_{\alpha} \). We claim that \( \{ \chi_\alpha / n_\alpha : \alpha \in U^0 \} \) is an equicontinuous, uniformly bounded set of functions. This is the case since

\[ |\text{Tr} (I - T_{\alpha})| \leq n_\alpha \| I - T_{\alpha} \|_\infty. \]

Further \( \| \chi_\alpha / n_\alpha \|_\infty \leq 1 \), and hence \( \{ \chi_\alpha / n_\alpha : \alpha \in U^0 \} \) is relatively compact, by the Arzelà-Ascoli theorem. Since the \( \{ \chi_\alpha / n_\alpha \} \) are orthogonal in \( L^2(G) \), either \( U^0 \) is finite or \( \{ \chi_\alpha / n_\alpha : \alpha \in U^0 \} \) has 0 as a uniform cluster point. This latter condition cannot happen, since \( \chi_\alpha(e) / n_\alpha = 1 \).

Let \( G \) be as above (that is, compact and nonabelian). We shall give the analogue to Corollary 7. Let the set \( \phi = \{ \phi_\alpha : \alpha \in \hat{G} \}, \) where \( \phi_\alpha \in B(C^{n_\alpha}) \) be such that \( \sup \{ \| \phi_\alpha \|_\infty : \alpha \in \hat{G} \} < \infty \). The set of all such \( \phi \) is denoted by \( B^\infty(\hat{G}) \). It is a Banach algebra under the norm \( \| \phi \|_\infty = \sup \{ \| \phi_\alpha \|_\infty : \alpha \in \hat{G} \} \) and under co-ordinatewise operations. Let

\[ \mathcal{E}_0(\hat{G}) = \{ \phi \in B^\infty(\hat{G}) : \lim_{\alpha \to \infty} \| \phi_\alpha \|_\infty = 0 \}. \]

For \( \mu \in M(G) \), the Fourier transform \( \hat{\mu} \) of \( \mu \) is a matrix-valued function, defined for \( \alpha \in \hat{G} \) by the relation

\[ \alpha \mapsto \hat{\mu}_{\alpha} = \int_G T_{\alpha}(x^{-1}) d\mu(x). \]

Note that \( \hat{\mu} \in B^\infty(\hat{G}) \). Thus for \( \mu \in M(G) \), let \( \| \mu \|_\infty = \sup \{ \| \hat{\mu}_{\alpha} \|_\infty : \alpha \in \hat{G} \} \). We define \( M_0(G) \) to be the set \( \{ \mu \in M(G) : \hat{\mu} \in \mathcal{E}_0(\hat{G}) \} \).

Let \( A \in B(X) \), where \( X \) is a finite-dimensional, complex inner-product space. We define the dual norm to \( \| \cdot \|_\infty \) by \( \| A \|_1 = \sup \{ |\text{Tr} (AB)| : \| B \|_\infty \leq 1 \} \). This norm can also be characterized by the condition \( \| A \|_1 = \text{Tr} (|A|) \). For \( \phi \in B^\infty(G) \), we put

\[ \| \phi \|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \| \phi_{\alpha} \|_1. \]

Let \( B^1(\hat{G}) = \{ \phi \in B^\infty(\hat{G}) : \| \phi \|_1 < \infty \} \). Then \( B^1(\hat{G}) \) is a Banach space under \( \| \cdot \|_1 \).

For \( \phi \in B^1(\hat{G}) \), let \( \text{Tr}(\phi) = \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr}(\phi_{\alpha}) \). For \( \psi \in B^1(G) \) and \( \phi \in B^\infty(G) \), we obtain the inequality \( |\text{Tr}(\phi \psi)| \leq \| \phi \|_\infty \| \psi \|_1 \).

We now define \( A(G) \), the Fourier algebra of \( G \), and we pair \( A(G) \) and \( M(G) \) to get the compact analogue of Corollary 7. Let \( A(G) \) be the set of \( f \in C(G) \) for which \( \hat{f} \in B^1(\hat{G}) \). We define a norm on \( A(G) \) by

\[ \| f \|_A = \| \hat{f} \|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \| \hat{f}_{\alpha} \|_1 < \infty. \]
Note that $A(G)$ is isomorphic to $L^1(\hat{G})$, because for each $\phi \in L^1(\hat{G})$, the function $f(x) = \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr} (\phi_\alpha T_\alpha(x))$ is in $A(G)$; further,

$$
\| f \|_\infty = \sup_{x \in G} \left| \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr} (\phi_\alpha T_\alpha(x)) \right| \leq \sum_{\alpha \in \hat{G}} n_\alpha \| \phi_\alpha \|_1 = \| \phi \|_1.
$$

We note that for $f \in A(G)$, $\| L(x)f \|_A = \| f \|_A$.

**Theorem 9.** Let $G$ be a compact (nonabelian) group, and let $\mu \in M(G)$. Then $x \mapsto L(x)\mu$ is continuous from $G$ to $(M(G), \| \cdot \|_\infty)$ if and only if $\mu \in M_0(G)$.

**Proof.** For $\mu \in M(G)$ and $f \in A(G)$, we define

$$
\langle f, \mu \rangle = \int_G f(t) d\mu(t) = \text{Tr} (\hat{\mu} h),
$$

where $h(t) = f(t^{-1})$. If $\hat{f}$ is defined by $\hat{f}(t) = f(t^{-1})$, then $\| \hat{f} \|_A = \| f \|_A$. Thus $\langle f, \mu \rangle = \text{Tr} (\hat{\mu} \hat{f})$. Let $A_k = \{ f \in A(G) : \| f \|_A < k \}$, and let $\mathcal{S}(A_k)$ be the topology on $M(G)$ of uniform convergence on the sets $A_k$. Since

$$
|\text{Tr} (\hat{\mu} \hat{f})| \leq \| \hat{\mu} \|_\infty \| \hat{f} \|_1 = \| \hat{\mu} \|_\infty \| f \|_A = \| \mu \|_\infty \| f \|_A,
$$

the topology $\mathcal{S}(A_k)$ is weaker than the $\| \cdot \|_\infty$-topology on $M(G)$. However, since $L^\infty(\hat{G})$ is identified with the dual space of $L^1(\hat{G})$ by $\psi \mapsto \text{Tr} (\phi\psi)$ for $\phi \in L^\infty(\hat{G})$ and $\psi \in L^1(\hat{G})$, $\mathcal{S}(A_k)$ is the same as the $\| \cdot \|_\infty$-topology on $M(G)$. Furthermore, $A_k$ is $L(x)$-invariant, since $\| L(x)f \|_A = \| f \|_A$ for $f \in A(G)$. We now apply Theorems 3 and 4. \[\blacksquare\]

We conclude now with the general case. We shall use the machinery developed by P. Eymard [2], and we shall follow his conventions in the use of $x$ in various formulae, where we used $x^{-1}$ in the compact and abelian cases discussed above.

Let $G$ be a locally compact group. Let $\Sigma$ denote the equivalence classes of the continuous unitary representations on $G$. For $\pi \in \Sigma$, let $\mathcal{H}_\pi$ denote the representation space. We define $\hat{\mu}$ to be a function on $\Sigma$ by $\pi \mapsto \hat{\mu}(\pi) = \int_G \pi(x) d\mu(x)$. For $\mathscr{F} \subset \Sigma$, let

$$
\| \mu \|_{\mathscr{F}} = \sup \{ \| \hat{\mu}(\pi) \|_\infty : \pi \in \mathscr{F} \},
$$

where $\| \hat{\mu}(\pi) \|_\infty$ denotes the operator norm on $\mathcal{H}_\pi$. We define $C^*(G)$ to be the completion of $L^1(G)$ in $\| \cdot \|_\Sigma$ (see [2, Section 1.14]). Let $\{ \rho \}$ denote the subset of $\Sigma$ containing just the left regular representation of $G$ on $L^2(G)$. Let $C^*_\rho(G)$ denote the completion of $L^1(G)$ in $\| \cdot \|_\rho$ (see [2, Section 1.16]).

For $\mu \in M(G)$, we let $\rho(\mu)$ denote the bounded operator on $L^2(G)$, defined by $h \mapsto \mu * h$ ($h \in L^2(G)$), with operator norm $\| \rho(\mu) \|_\rho$. Let $\mathcal{B}(L^2(G))$ denote the set of bounded operators on $L^2(G)$. Then $C^*_\rho(G)$ can be identified with the closure of $\rho(L^1(G)) = \{ \rho(f) : f \in L^1(G) \}$ in $\mathcal{B}(L^2(G))$. If $G$ is abelian, then $C^*_\rho(G) = C_0(G)$. If $G$ is compact, then $C^*(G) = C_0(\hat{G})$. 

Let $\text{VN}(G)$ denote the von Neumann subalgebra of $\mathcal{B}(L^2(G))$ generated by the left translation operators (see [2, Section 3.9]). For $\mu \in M(G)$, we have that $\rho(\mu) \in \text{VN}(G)$. Further, $C^*_p(G) \subset \text{VN}(G)$. If $G$ is abelian, then $\text{VN}(G) = L^\infty(\hat{G})$. If $G$ is compact, then $\text{VN}(G) = L^\infty(\hat{G})$.

**Definition.** $M_0(G) = \{ \mu \in M(G); \rho(\mu) \in C^*_p(G) \}$.

Let $B(G)$ denote the linear subspace of $C^B(G)$ generated by the continuous positive-definite functions. Then $B(G)$ can be identified with the dual space of $C^*(G)$ (see [2, Section 2.2]). For $f \in B(G)$, let $\|f\|_B$ denote the norm of $f$ as a linear functional on $C^*(G)$. Finally, let $A(G)$ be the closed subspace of $B(G)$ generated by the continuous positive-definite functions with compact support (see [2, Section 3.4]). If $G$ is abelian, then $A(G) = L^1(\hat{G})^\times$. If $G$ is compact, then our previous definitions and those of Eymard are consistent. We have the inclusion $A(G) \subset C^B_{\text{ru}}(G)$, since $A(G) \subset C_0(G)$. We let $A_k = \{ f \in A(G); \|f\|_B < k \}$. Now for $f \in A(G)$, $\|L(x)f\|_B = \|f\|_B$; hence each $A_k$ is $L(x)$-invariant. We pair $A(G)$ and $M(G)$ by the relation

$$\langle f, \mu \rangle = \int_G f(t)d\mu(t) \quad (f \in A(G) \text{ and } \mu \in M(G)).$$

Let $\mathcal{F}(A_k)$ be the topology on $M(G)$ of uniform convergence on the sets $A_k$. We wish to apply Theorems 3 and 4 as we did in Theorem 9. To do this, it remains only to observe that $\text{VN}(G)$ can be identified as the dual space of $A(G)$ (see [2, Section 3.10]), and for $\mu \in M(G)$, the identification is given by the relation

$$f \mapsto \int_G f(x)d\mu(x) = \langle f, \mu \rangle,$$

where $f \in A(G)$. It follows now by Theorems 3 and 4 that $x \mapsto L(x)\mu$ is continuous from $G$ to $(M(G), \| \cdot \|_\rho)$ if and only if $\rho(\mu) \in \rho(L^1(\hat{G}))$ (the closure in $\mathcal{B}(L^2(G))$).

Hence we have the following result.

**THEOREM 10.** Let $G$ be a locally compact group. Let $\mu \in M(G)$. Then $x \mapsto L(x)\mu$ is continuous from $G$ to $(M(G), \| \cdot \|_\rho)$ if and only if $\mu \in M_0(G)$.

**THEOREM 11.** Suppose $A \subset C^B_{\text{ru}}(G)$ has the further property that $A$ is dense in $L^1(|x|)$ for each $\mu \in M(G)$, and that for each $f \in A$ we have inclusions $fA_k \subset CA_k \setminus (k = 1, 2, \cdots)$, where the constants $C$ and $k'$ depend on $f$ and on $k$. Then $L^1(\hat{G})^A$ is a band; in other words, if $\mu \in L^1(\hat{G})^A$ and $\nu \ll \mu$, then $\nu \in L^1(\hat{G})^A$.

**Proof.** Let $\mu \in L^1(\hat{G})^A$ and $\nu \ll \mu$; then $d\nu = g d\mu$, for some Borel function $g \in L^1(|x|)$. Now there exist functions $f_m \in A$ ($m = 1, 2, \cdots$) such that

$$\int_G |f_m - g|d|\mu| < 1/m,$$

that is, $\|f_m d\mu - d\nu\|_{M(G)} \to 0$ as $m \to \infty$. We claim that each $f_m d\mu$ belongs to $L^1(\hat{G})^A$. For if $\{g_n\} \subset L^1(G)$ and $g_n \to \mu$ in $\mathcal{F}(A_k)$, then $f_m g_n \to f_m d\mu$ (note that $f_m g_n \in L^1(G)$). In fact, for each $k$, we have the relations
\[ \tau_k(f_m g_n - f_m d\mu) = \sup \left\{ \left| \int_G \phi(x)f_m(x)[g_n(x)dx - d\mu(x)] \right| : \phi \in A_k \right\} \]

\[ \leq C \sup \left\{ \left| \int_G \phi(x)[g_n(x)dx - d\mu(x)] \right| : \phi \in A_{k'} \right\} = C \tau_{k'}(g_n - d\mu), \]

where \( C \) and \( k' \) depend on \( k \) and \( f_m \). Thus \( \tau_k(f_m g_n - f_m d\mu) \to 0 \), and \( f_m d\mu \in L^1(G)^A \).

Since \( \mathcal{F}(A_k) \)-closed sets are closed in the measure norm topology \( (\sup \{ \| \phi \|_\infty : \phi \in A_k \} < \infty) \), we have that \( \nu \in L^1(G)^A \). \( \blacksquare \)

**Corollary 12.** For every locally compact group \( G \), \( M_0(G) \) is a band.

**Proof.** Let \( A = A(G) \) as before, and recall that \( A(G) \) is a dense subalgebra of \( C_0(G) \) (for the locally compact case, see [2, Section 3.4]). \( \blacksquare \)

**References**


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