

### A characterization of quotient algebras of $L^1(G)$

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Let  $G$  be a locally compact Abelian group;  $\Gamma$  the dual group of  $G$ ;  $C_0(\Gamma)$  the algebra of continuous functions on  $\Gamma$  which vanish at infinity;  $C^B(\Gamma)$  the continuous, bounded functions on  $\Gamma$ ;  $M(G)$  the algebra of bounded Borel measures on  $G$ ;  $L^1(G)$  the algebra of absolutely continuous measures; and  $M(G)^\wedge$  the algebra of Fourier-Stieltjes transforms.

Let  $E$  denote a compact subset of  $\Gamma$ , and let  $A(E)$  be the set of all functions on  $E$  which are restrictions to  $E$  of functions belonging to  $L^1(G)^\wedge$ . The norm in  $A(E)$  is the quotient norm; i.e.

$$\|f\|_{A(E)} = \inf \{ \|g\|_{L^1(G)} : g \in L^1(G) \text{ and } \hat{g}|_E = f \}.$$

For  $f \in C(E)$ , let

$$\|f\|_{B(E)} = \sup \left\{ \left| \int_E f d\lambda \right| : \lambda \in M(E) \text{ and } \|\lambda^\wedge\|_\infty \leq 1 \right\}.$$

Let  $B(E)$  consist of those functions  $f \in C(E)$  for which  $\|f\|_{B(E)}$  is finite. Now

$$A(E) \subset B(E) \subset C(E) \text{ and } \|f\|_{C(E)} \leq \|f\|_{B(E)} \leq \|f\|_{A(E)}, \text{ for } f \in A(E).$$

**THEOREM 1.**  $B(E)$  is a commutative, semi-simple, self-adjoint Banach algebra with unit under point-wise operations.

*Proof.* Katznelson and McGehee have noted that  $B(E)$  is a Banach space(3). Let  $\{f_n\}$  be a Cauchy sequence in  $B(E)$  and  $f$  its limit point in  $C(E)$ . Let  $0 \neq \lambda \in M(E)$ , then

$$\left| \int_E (f - f_n) d\lambda \right| \leq \left| \int_E (f - f_m) d\lambda \right| + \|f_n - f_m\|_{B(E)} \|\lambda^\wedge\|_\infty.$$

Let  $\epsilon > 0$ , and  $N$  be such that if  $n, m \geq N$ , then  $\|f_n - f_m\|_{B(E)} < \frac{1}{2}\epsilon$ . Pick  $m \geq N$  such that  $\|f - f_m\|_{C(E)} < \epsilon/(2\|\lambda\|)$ . Thus  $\|f - f_n\|_{B(E)} < \epsilon$  for all  $n \geq N$ . Hence  $f$  is the limit point of  $\{f_n\}$  in  $B(E)$ .

$B(E)$  is an algebra since for  $\lambda \neq 0$ ,

$$\begin{aligned} \frac{\left| \int_E fg d\lambda \right|}{\|\lambda^\wedge\|_\infty} &= \frac{\left| \int_E fg d\lambda \right|}{\|(gd\lambda)^\wedge\|_\infty} \frac{\|(gd\lambda)^\wedge\|_\infty}{\|\lambda^\wedge\|_\infty} \leq \|f\|_{B(E)} \frac{\|(gd\lambda)^\wedge\|_\infty}{\|\lambda^\wedge\|_\infty} \\ &\leq \|f\|_{B(E)} \frac{\sup_{\gamma \in G} \left| \int \gamma g d\lambda \right|}{\|\lambda^\wedge\|_\infty} \\ &\leq \|f\|_{B(E)} \|g\|_{B(E)} \frac{\sup_{\gamma \in G} \|(\gamma d\lambda)^\wedge\|_\infty}{\|\lambda^\wedge\|_\infty} \\ &= \|f\|_{B(E)} \|g\|_{B(E)}. \end{aligned}$$

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Thus for  $f, g \in B(E)$ ,  $\|fg\|_{B(E)} \leq \|f\|_{B(E)}\|g\|_{B(E)}$ . Similarly,  $B(E)$  is self-adjoint since for  $\lambda \neq 0$ ,

$$\frac{\left| \int \bar{f} d\lambda \right|}{\|\lambda^\wedge\|_\infty} = \frac{\left| \int f d\bar{\lambda} \right|}{\|\lambda^\wedge\|_\infty} = \frac{\left| \int f d\bar{\lambda} \right|}{\|(\bar{\lambda})^\wedge\|_\infty} \quad \blacksquare$$

It is natural to ask when is  $A(E) = B(E)$  and when is  $B(E) = C(E)$ . If  $B(E) = C(E)$ , then the closed graph theorem implies that  $E$  is a Helson set; i.e.  $A(E) = C(E)$ . Thus  $B(E) = C(E)$  if and only if  $E$  is a Helson set. (This was pointed out to the author by C. Dunkl.) If  $E$  is a compact subset of  $\Gamma$  which is of uniform positive measure, then  $A(E) = B(E)$  (see (7) for a stronger result). Katznelson and McGehee(3) have given examples of compact sets such that  $A(E) \neq B(E)$  (see also (8)). In all examples known to the author,  $A(E)$  is, however, a closed subspace of  $B(E)$  from which it follows that the  $A(E)$  and  $B(E)$  norms are equivalent on  $A(E)$ . For such  $E$  we characterize  $A(E)$  in  $C(E)$ . Our proof yields a characterization of the  $B(E)$ -closure of  $A(E)$  in  $C(E)$ .

We pair  $\langle A(E), M(E) \rangle$  by  $\langle f, \mu \rangle = \int_E f d\mu$ . If  $\langle f, \mu \rangle = 0$ , for all  $\mu \in M(E)$ , in particular for point measures, then  $f = 0$ . If  $\langle f, \mu \rangle = 0$  for all  $f \in A(E)$ , then  $\mu = 0$  since  $A(E)$  is sup-norm dense in  $C(E)$ . Let  $w$  denote the weak topology on  $M(E)$  from this pairing; i.e.  $\lambda_x \xrightarrow{\alpha} 0$  in  $w$  if and only if  $\int_E f d\lambda_x \xrightarrow{\alpha} 0$  for all  $f \in A(E)$ . Finally, let  $B_n^\wedge$  denote the set of  $\mu \in M(E)$  such that  $\|\mu^\wedge\|_\infty \leq n$ .

**THEOREM 2.** *Let  $A(E)$  be a closed subspace of  $B(E)$ . For  $f \in C(E)$ , the following are equivalent:*

- (A)  $f \in A(E)$ .
- (B)  $\mu \rightarrow \int_E f d\mu$  is  $w$ -continuous on  $B_n^\wedge$ .

*Proof.* We use a technique introduced in (5). Now  $B_n^\wedge$  is a convex circled set in  $M(E)$ . It is  $w$ -bounded since

$$|\langle f, \mu \rangle| \leq \|\mu^\wedge\|_\infty \|f\|_{A(E)}.$$

It is  $w$ -closed since the restrictions of the continuous characters to  $E$  are in  $A(E)$ . Let  $T$  denote the topology on  $A(E)$  of uniform convergence on the sets  $B_n^\wedge$ . Our hypothesis asserts that  $T$  is the  $B(E)$ -norm topology.

The Grothendeick completeness theorem ((4), p. 149) yields the result since  $A(E)$  is  $B(E)$ -norm closed.  $\blacksquare$

If  $A(E)$  is not a closed subspace of  $B(E)$ , then the above proof yields a characterization of the  $B(E)$ -closure of  $A(E)$  in  $C(E)$ .

Let  $E$  be such that the set  $M_0(E)$ , consisting of  $\mu \in M(E)$  such that  $\mu \in C_0(G)$ , has the property that for  $f \in C(E)$  if  $\int_E f d\mu = 0$  for all  $\mu \in M_0(E)$ , then  $f = 0$ . We call  $E$  a Tauberian set. For example, if  $E$  is of uniform positive measure, then  $E$  is Tauberian.

MAIN THEOREM 3. Let  $E$  be a compact, Tauberian set. Then  $A(E) = B(E)$ , and for  $f \in C(E)$ , the following are equivalent:

(A)  $f \in A(E)$ .

(C) If  $\mu_n \in M(E)$ ,  $\|\mu_n^\wedge\|_\infty \leq 1$ , and  $\mu_n^\wedge \xrightarrow{n} 0$  pointwise on  $G$ , then  $\int_E f d\mu_n \xrightarrow{n} 0$ .

Proof. Let  $f \in B(E)$ . Consider the linear functional on  $M_0(E)^\wedge$  defined by  $\mu^\wedge \rightarrow \int_E f d\mu$ .

Since  $f \in B(E)$ , the linear functional is sup-norm continuous and has an extension to  $C_0(G)$ . Thus there is  $\lambda \in M(G)$  such that

$$\int_E f d\mu = \int_G \mu^\wedge d\lambda = \int_E \lambda^\wedge d\mu,$$

for all  $\mu \in M_0(E)$ . Since  $E$  is Tauberian,  $f = \lambda^\wedge|_E$  and hence  $f \in A(E)$ .

Assume (A). Thus there is  $g \in L^1(G)$  such that  $\hat{g}|_E = f$ . Thus for  $\{\mu_n\}$  as in (C), we have that

$$\int_E f d\mu_n = \int_E \hat{g} d\mu_n = \int_\Gamma \hat{g} d\mu_n = \int_G \mu_n^\wedge dg \xrightarrow{n} 0$$

by the Lebesgue dominated convergence theorem.

Assume (C). If  $A(E)$  is separable, then the  $w$  topology on  $B_n^\wedge$  is metrizable and it follows easily that  $\mu \rightarrow \int_E f d\mu$  is  $w$ -continuous on  $B_n^\wedge$ .

Since we are following our proof from ((5), p. 329), we just outline this proof.

Represent the continuous bounded functions on  $G$ ,  $C^B(G)$ , as operators on  $C_0(G)$  by  $T_f(g) = fg$ ,  $f \in C^B(G)$ ,  $g \in C_0(G)$ . Let  $WO$  denote the weak operator topology on  $C^B(G)$  via this representation and  $SO$  the strong operator topology, (2). The  $SO$  topology is Buck's 'strict' topology, and it agrees with the compact-open topology on sup-norm bounded sets, ((1), p. 98). Viewing  $M(E)$  as a subspace of  $C^B(G)$  via the Fourier-Stieltjes transformation induces the  $WO$  and  $SO$  topologies on  $M(E)$ . It suffices by

Theorem 2, to show that  $\mu \rightarrow \int_E f d\mu$  is  $w$ -continuous on  $B_n^\wedge$ .

Now  $\mu_\alpha \xrightarrow{\alpha} 0$  in  $w$  if and only if  $\int_E f d\mu_\alpha \xrightarrow{\alpha} 0$  for all  $f \in A(E)$  if and only if  $\int_E \lambda^\wedge d\mu_\alpha \xrightarrow{\alpha} 0$  for all  $\lambda \in M(G)$  since  $E$  is compact. It follows that the weak topology

is equivalent to the  $WO$ -topology on  $B_n^\wedge$ . Thus we show that  $\mu \rightarrow \int_E f d\mu$  is  $WO$ -continuous on  $B_n^\wedge$ . This is equivalent to showing that  $\mu \rightarrow \int_E f d\lambda$  is  $SO$ -continuous on  $B_n^\wedge$ .

From (C) it follows that  $\mu \rightarrow \int_E f d\mu$  is  $SO$ -continuous on the sets

$$C_n^\wedge = \{\lambda \in M_0(E) : \|\lambda^\wedge\|_\infty \leq n\}$$

and hence  $WO$ -continuous on  $C_n^\wedge$  and hence  $w$ -continuous on  $C_n^\wedge$ . We extend  $\mu \rightarrow \int_E f d\mu$  from  $M_0(E)$  to  $M(E)$ , ((4), p. 49). The extended linear functional has the form  $\mu \rightarrow \int_E g d\mu$ ,  $g \in A(E)$ , and is  $w$ -continuous on  $B_n^\wedge$ . Since  $E$  was chosen to be a Tauberian set,  $f = g$ . ■

The technique used to characterize  $A(E)$  in  $C(E)$  can be used to characterize other linear spaces. Let  $M_d(G)$  denote the discrete measures on  $G$ . We may pair  $M_d(G)^\wedge$  and  $M(\Gamma)$  by  $\langle \mu^\wedge, \lambda \rangle = \int_\Gamma \mu^\wedge d\lambda$ . Let  $\tau$  denote the weak topology on  $M(\Gamma)$  from this pairing. Let  $B_n = \{\lambda \in M(\Gamma) : \|\lambda\| \leq n\}$ . Let  $AP(\Gamma)$  denote the algebra of continuous almost periodic functions on  $\Gamma$ .

**THEOREM 4.** *Let  $G$  be a locally compact Abelian group. For  $f \in C^B(\Gamma)$  the following are equivalent:*

(A)  *$f$  is almost periodic.*

(B)  $\lambda \rightarrow \int_\Gamma f d\lambda$  is  $\tau$ -continuous on  $B_n$ .

(C) *If  $\{\lambda_\alpha\} \subset B_n$  and  $\lambda_\alpha \xrightarrow{\alpha} 0$  pointwise on  $G$ , then  $\int_\Gamma f d\lambda_\alpha \xrightarrow{\alpha} 0$ .*

*Proof.* Now  $B_n$  is a convex circled set in  $M(\Gamma)$ . It is  $\tau$ -bounded since

$$|\langle \mu^\wedge, \lambda \rangle| \leq \|\mu^\wedge\|_\infty \|\lambda\|.$$

It is  $\tau$ -closed since

$$\sup \left\{ \left| \int_\Gamma \mu^\wedge d\lambda \right| : \mu \in M_d(G), \|\mu^\wedge\|_\infty \leq 1 \right\} = \|\lambda\|.$$

Let  $T'$  denote the topology on  $M_d(G)^\wedge$  of uniform convergence on the sets  $B_n$ . Since  $\sup \left\{ \left| \int_\Gamma \mu^\wedge d\lambda \right| : \lambda \in M(\Gamma), \|\lambda\| \leq 1 \right\} = \|\mu^\wedge\|_\infty$ ,  $T'$  is the sup-norm topology.

The Grothendieck completeness theorem yields that (A) is equivalent to (B) since the sup-norm completion of  $M_d(G)^\wedge$  in  $C^B(\Gamma)$  is  $AP(\Gamma)$ .

That (B) is equivalent to (C) is immediate. ■

The previous result is due to R. Edwards ((9), p. 254). His method is based on the bipolar theorem. Finally, we give a simple proof of a known result on almost periodic functions ((10), p. 449).

**THEOREM 5.** *Let  $f$  be a continuous almost periodic function on the locally compact, non-compact, Abelian group  $\Gamma$ . Let  $K$  be a compact subset of  $\Gamma$ . Then*

$$\sup \{|f(x)| : x \in K\} \leq \sup \{|f(x)| : x \notin K\}.$$

*Proof.* Since  $\Gamma$  is non-compact, we let  $\Gamma'$  denote the Bohr compactification of  $\Gamma$  ((6), p. 30). Let  $\beta: \Gamma \rightarrow \Gamma'$  denote the natural continuous isomorphism of  $\Gamma$  into  $\Gamma'$ . Since  $K$  is compact and  $\beta$  is continuous,  $\beta(K)$  is also compact in  $\Gamma'$ .

Suppose by way of contradiction that

$$\alpha = \sup \{|f(x)| : x \in K\} > \gamma = \sup \{|f(x)| : x \notin K\}.$$

Extend  $f$  to a continuous function,  $f'$ , on  $\Gamma'$ . Since  $\beta(K)$  is closed in  $\Gamma'$ ,  $|f'(y)| \leq \gamma$  for  $y \in \Gamma' \setminus \beta(K)$ . Thus  $K$  contains an open subset  $U$  of  $\Gamma$ . It follows that  $\beta$  is a homeomorphism on  $U$ . Since  $\beta$  is a homomorphism,  $\beta$  is a homeomorphism on all of  $\Gamma$ . Thus  $\Gamma$  is compact. This contradiction completes the proof. ■

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