In a comprehensive paper by Riggs et al. (1978) the authors analyse the performances of numerous estimators for the regression slope in the measurement error model with positive measurement error variances $\sigma_X^2 > 0$ for $X$ and $\sigma_Y^2 > 0$ for $Y$. In particular, using a Monte Carlo simulation, the authors demonstrate that the adjusted geometric mean estimator of Madansky (1959, Equation 4, p. 179), which requires knowledge of both $\sigma_X^2$ and $\sigma_Y^2$, performs “much worse than” the maximum likelihood estimator in the normal structural measurement error model which requires only knowledge of the ratio $\kappa = \sigma_Y^2 / \sigma_X^2$. The second moment estimator, $\beta_1^{kap}$, coincides with the maximum likelihood estimator in the normal structural measurement error model (Madansky (1959)). In practice $\kappa$ has to be estimated by $\bar{r}$. In this paper, we show that the bias of $\beta_1^{kap}$ is not only dependent on the magnitude of the difference between $\kappa$ and $\bar{r}$ but also on the magnitude of $\bar{r}\sigma_Y^2 - \sigma_X^2$. We use a fourth moment estimator to smooth the jump discontinuity in the estimator of Copas (1972) as described in ODriscoll and Ramirez (2011) and use this estimator to find estimates for each error variance $\sigma_X^2$ and $\sigma_Y^2$. Our Monte Carlo simulations show that the adjusted geometric mean estimator of Madansky performs much better than the ordinary least squares estimators OLS($y|x$) and OLS($x|y$) when the error variances are strictly positive and performs equally as well as the geometric estimator, $\beta_1^{gm}$, and the perpendicular estimator of Adcock (1878), $\beta_1^{per}$, with $\kappa = 1$.

**Keywords:** Moment estimation; measurement errors; errors in variables.

**Introduction**
With ordinary least squares OLS($y|x$) regression we have data 
\[ \{ (x_1, Y_1|X_1 = x_1), \ldots, (x_n, Y_n|X_n = x_n) \}, \]
and we minimize the sum of the squared vertical errors to find the best-fit line $y = h(x) = \beta_0 + \beta_1 x$, where it is assumed that the independent or causal variable $X$ is measured without error. In this paper it is assumed that $X$ and $Y$ are random variables with respective finite variances $\sigma_X^2$ and $\sigma_Y^2$, finite fourth moments and have the linear relationship $Y = \beta_0 + \beta_1 X$. It is also assumed that the $x_i$ are a random sample from the random variable $X$. In the chapter of Measurement Error Model we outline the assumptions of the measurement error model and give second and fourth order moment equations. In the next chapter we give an expression for the asymptotic bias of the second moment estimator. We show that this bias is dependent on the magnitude of the error variance $\sigma_\delta^2$ associated with the $X$ variable and illustrate this result in Table 1 using a Monte Carlo simulation. We introduce a fourth order slope estimator and suggest that this estimator be used in the second order moment equations to estimate $\sigma_\delta^2$ and $\sigma_\tau^2$. When $\kappa = \sigma_\tau^2 / \sigma_\delta^2$, the maximum likelihood estimator $\hat{\beta}_1^{kap} = \tilde{\beta}_1(\kappa)$ (see for example, Madansky (1959, Equation 3)) has been dubbed by Riggs (1978, p. 1320) as the Properly Weighted Perpendicular Least Squares Estimator. The geometric mean of Madansky’s estimators from his Equations 1 and 2 is given in Equation 4 which we have dubbed as Madansky’s Adjusted Geometric Mean Estimator and is denoted by $\hat{\beta}_1^{agm}$. Using a second Monte Carlo simulation, we investigate the performance of each of the five estimators estimators $\hat{\beta}_1^{ver}, \hat{\beta}_1^{hor}, \hat{\beta}_1^{per}, \hat{\beta}_1^{agm}$ and $\hat{\beta}_1^{agm}$. In the next chapter we present two applications of $\hat{\beta}_1^{agm}$ and compare its performance to the results in the literature. In the final chapter of this paper we conclude that $\hat{\beta}_1^{agm}$ performs equally as well as $\hat{\beta}_1^{ver}$ and $\hat{\beta}_1^{agm}$ and much better than the ordinary least squares estimators $\hat{\beta}_1^{ver}$ and $\hat{\beta}_1^{hor}$.

The Measurement Error Model

The observed data $\{(x_i, y_i), 1 \leq i \leq n\}$ are subject to error by $x_i = X_i + \delta_i$ and $y_i = Y_i + \tau_i$, where it is also assumed that $\delta$ is $N(0, \sigma_\delta^2)$, $\tau$ is $N(0, \sigma_\tau^2)$, $\text{Cov}(\delta_i, \delta_j) = 0$, $i \neq j$, $\text{Cov}(\tau_i, \tau_j) = 0$, $i \neq j$ and $\text{Cov}(\delta_i, \tau_j) = 0$, for all $i$ and $j$. Let

\[
 s_{xx} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2, \quad s_{yy} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 \quad \text{and} \quad s_{xy} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) .
\]

From Gillard and Iles (2009) it follows that second moment equations are

\[
s_{xx} = \sigma_X^2 + \sigma_\delta^2, \quad s_{yy} = \beta_1^2 \sigma_X^2 + \sigma_\tau^2, \quad s_{xy} = \beta_1 \sigma_X^2
\]

which yield the second moment estimators
Fourth moment equations are

\[ s_{xxy} = \beta_1 \mu_X + 3 \beta_1 \sigma_X^2 \sigma_\epsilon^2; \quad s_{xy} = \beta_1^3 \mu_X + 3 \beta_1 \sigma_X^2 \sigma_\epsilon^2. \]  

If the ratio of the error variances \( \kappa = \sigma_\epsilon^2 / \sigma_\delta^2 \) is assumed finite, then the second moment estimator for the regression slope is

\[ \hat{\beta}_1 = \frac{s_{yy} - \kappa s_{xx}}{s_{xy}} + \sqrt{\left( s_{yy} - \kappa s_{xx} \right)^2 + 4 \kappa s_{xy}^2} \]

which coincides with the maximum likelihood estimator of the regression slope in the normal structural measurement error model (see Madansky (1959) and Davidov (2005)). If \( \kappa = 1 \) in Eqn. (4), then the moment estimator (often called the Deming (1943) Regression Estimator) is equivalent to the perpendicular estimator, \( \hat{\beta}_1^{per} \), first introduced by Adcock (1878). Similarly, if \( \kappa = 0 \) in Eqn. (4), then the moment estimator reduces to \( OLS(X|Y) \), \( \hat{\beta}_1^{hor} \), and if \( \kappa = \infty \) then the moment estimator reduces to \( OLS(Y|X) \), \( \beta_1^{per} \).

Since the second sample moments converge in probability to their expectations (see Davidov (2005) and Mamun et al. (2013)), it follows from (1) that

\[ \frac{s_{yy} - \kappa s_{xx}}{s_{xy}} = \beta_1 - \kappa / \beta_1 \quad \text{and} \quad \frac{s_{xx}}{s_{xy}} = \frac{1 + \sigma_\delta^2 / \sigma_\epsilon^2}{\beta_1}, \]

where \( \sigma_\delta^2 / \sigma_\epsilon^2 \) is the noise to signal ratio of the model.

If the researcher knows the true error \( \kappa = \sigma_\epsilon^2 / \sigma_\delta^2 \), then

\[ \hat{\beta}_1(\kappa) \xrightarrow{p} 0.5 \left( (\beta_1 - \kappa / \beta_1) + \sqrt{(\beta_1 - \kappa / \beta_1)^2 + 4\kappa} \right) = \beta_1 \]

and there are no asymptotic bias problems. We will discuss the more realistic situation when \( \kappa \) is an unknown parameter and must be estimated by \( \bar{\kappa} \).

The Empirical Bias of the Second Moment Estimator for an Incorrect Choice of \( \kappa \)

**Empirical Bias**

In practice, the researcher estimates \( \kappa \) by \( \bar{\kappa} \) with error \( e = \kappa - \bar{\kappa} \neq 0 \). To develop an expression for the asymptotic bias \( E(\tilde{\beta}_1(\bar{\kappa}) - \beta_1) \), we recall Eqn. (4) and write

\[ E(\tilde{\beta}_1(\bar{\kappa}) - \beta_1) = E(\tilde{\beta}_1(\bar{\kappa}) - \tilde{\beta}_1(\kappa)) + E(\tilde{\beta}_1(\kappa) - \beta_1) = E(\tilde{\beta}_1(\bar{\kappa}) - \tilde{\beta}_1(\kappa)). \]

We define the **empirical bias** in using \( \bar{\kappa} \) to estimate \( \kappa \) as \( \text{empbias}(\bar{\kappa}:\kappa) = \tilde{\beta}_1(\bar{\kappa}) - \tilde{\beta}_1(\kappa) \), which in terms of \( \{s_{xx}, s_{yy}, s_{xy}\} \) is

\[ \frac{1}{2s_{xy}} \left( -s_{xx}(\bar{\kappa} - \kappa) + \sqrt{(s_{yy} - \bar{\kappa}s_{xx})^2 + 4\bar{\kappa}s_{xy}^2} - \sqrt{(s_{yy} - \kappa s_{xx})^2 + 4\kappa s_{xy}^2} \right). \]
The empirical bias is an estimate of the error that occurs in $\hat{\beta}_i(\kappa)$ as a result of using $\bar{\kappa}$ for the unknown error ratio $\kappa$. In our simulation study we record estimates for both $E_{bias} = E(\hat{\beta}_i(\bar{\kappa}) - \beta_i)$ and $E_{emp} = E(\hat{\beta}_i(\bar{\kappa}) - \hat{\beta}_i(\kappa))$ in Table 1. They are, as expected, nearly equal values, and are both close to the theoretical values for the bias, $bias(\bar{\kappa} : \kappa)$.

With $\varepsilon = \kappa - \bar{\kappa} \neq 0$ and $\theta = 1+\sigma_X^2/\sigma_Y^2$, the bias, $bias(\bar{\kappa} : \kappa)$, in terms of $\{\beta_i, \varepsilon, \kappa, \theta\}$ is then

$$
-0.5 \left( \beta_1 + \frac{\kappa}{\beta_1} + \frac{\theta \varepsilon}{\beta_1} \right) + 0.5 \left( \beta_1 + \frac{\kappa}{\beta_1} - \frac{\theta \varepsilon}{\beta_1} \right) \sqrt{1 + \frac{4\varepsilon \left( 1 + \frac{\theta \varepsilon}{\beta_1} \right)}{(\beta_1 + \frac{\kappa}{\beta_1} - \frac{\theta \varepsilon}{\beta_1})^2}}. \quad (7)
$$

We note that $bias(\bar{\kappa} : \kappa)= 0$ only when $\varepsilon = \kappa - \bar{\kappa} = 0$ as proven by Lindley, D., El-Sayyad, M. (1968).

**Series Expansion for the Bias**

The series expansion, $serbias(\bar{\kappa} : \kappa)$, of the bias may be written in terms of $\varepsilon$ as

$$
\frac{\epsilon \beta_1 (\theta - 1)}{\beta_1^2 + \kappa} \left( 1 - \frac{\epsilon (\beta_1^2 + \kappa \theta)}{(\beta_1^2 + \kappa)^2} - \frac{\epsilon^2 (-\kappa \theta + \theta \beta_1^2 - 2 \beta_1^2)(\beta_1^2 + \kappa \theta)}{(\beta_1^2 + \kappa)^4} \right) + O(\varepsilon^4). \quad (8)
$$

Since

$$
\frac{\epsilon \beta_1 (\theta - 1)}{\beta_1^2 + \kappa} = \frac{(\bar{\kappa} - \kappa)\sigma_X^2}{(\beta_1 + \frac{\kappa}{\beta_1}) \sigma_Y^2},
$$

Equation (8) shows that $bias(\bar{\kappa} : \kappa)$ is not alone dependent on the magnitude of $(\bar{\kappa} - \kappa)$, but is also dependent on the magnitude of $\sigma_Y^2$; that is, the magnitude of the bias is dependent on the magnitude of the difference $\bar{\kappa}\sigma_Y^2 - \sigma_X^2$, and our claim in the Abstract is justified.

**Monte Carlo Simulation**

We used Minitab for our simulation study setting the number of runs $N = 5000$, $\bar{\kappa} = 1$ and the sample size $n = 50$. The $X$ data was generated from a normal $\beta_1 = 1$ and $\beta_0 = 0$. For the measurement error model, we used normal errors with mean equal to zero and variances $\{\sigma_X^2, \sigma_Y^2\}$ varying over $\{1, 2, 3, 4, 9\}$. Typical values for the bias, $bias(\bar{\kappa} : \kappa)$ from Equation (7), and the Third Order series approximation, $serbias(\bar{\kappa} : \kappa)$ from Equation (8), are shown in Table 1. We note that the Third Order approximation yields, as expected, values close to the theoretical values for the bias, $bias(\bar{\kappa} : \kappa)$. We also record the results for the asymptotic bias $E(\hat{\beta}_i(\bar{\kappa}) - \beta_i)$, and the estimated empirical bias $E(\hat{\beta}_i(\bar{\kappa}) - \hat{\beta}_i(\kappa))$. 

150
An Investigation of the Performance of Five Different Estimators in the Measurement Error Regression Model

Table 1

<table>
<thead>
<tr>
<th>$\sigma_\epsilon^2$</th>
<th>$\sigma_\delta^2$</th>
<th>$\kappa$</th>
<th>$\sigma_\epsilon^2 - \sigma_\delta^2$</th>
<th>$E_{bias}$</th>
<th>$E_{emp}$</th>
<th>bias($\kappa : \kappa$)</th>
<th>serbias($\kappa : \kappa$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>0.333</td>
<td>-6.0</td>
<td>-0.0276</td>
<td>-0.0303</td>
<td>-0.0296</td>
<td>-0.0330</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.500</td>
<td>-2.0</td>
<td>-0.0085</td>
<td>-0.0101</td>
<td>-0.0100</td>
<td>-0.0103</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.333</td>
<td>-2.0</td>
<td>-0.0093</td>
<td>-0.0102</td>
<td>-0.0100</td>
<td>-0.0112</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.500</td>
<td>-1.0</td>
<td>-0.0046</td>
<td>-0.0051</td>
<td>-0.0050</td>
<td>-0.0052</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.000</td>
<td>1.0</td>
<td>0.0048</td>
<td>0.0051</td>
<td>0.0050</td>
<td>0.0048</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3.000</td>
<td>2.0</td>
<td>0.0109</td>
<td>0.0103</td>
<td>0.0100</td>
<td>0.0088</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2.000</td>
<td>2.0</td>
<td>0.0100</td>
<td>0.0102</td>
<td>0.0100</td>
<td>0.0097</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3.000</td>
<td>6.0</td>
<td>0.0321</td>
<td>0.0313</td>
<td>0.0304</td>
<td>0.0266</td>
</tr>
</tbody>
</table>

The rows of Table 1 are sorted in ascending order of the theoretical bias, $bias(\kappa : \kappa)$ displayed in Column 7. Column 8 shows that our approximation, serbias($\kappa : \kappa$), is a good estimate for bias($\kappa : \kappa$). Columns 5 and 6 also show that our simulation study produced very good results for $E(\hat{\beta}_1(\kappa) - \beta_1)$, and $E E(\hat{\beta}_1(\kappa) - \beta_1(\kappa))$.

We make the following observations. Firstly, with $\kappa = 1$, the ranking for the bias concurs with the ranking of the differences in the error variances $\sigma_\epsilon^2 - \sigma_\delta^2$ but does not concur with the ranking for $\kappa = \sigma_\epsilon^2 / \sigma_\delta^2$ in terms of its closeness to $\kappa$; that is, the magnitude of the bias for the MLE estimator $\hat{\beta}_1(\kappa)$ is not monotone in $\kappa$. Secondly, for equal $\kappa = 3/1$ in Row 6 and $\kappa = 9/3$ in Row 8, the respective biases 0.0101 and 0.0304 are approximately proportional to the respective differences of the error variances 2 and 6.

Since the asymptotic bias $E(\hat{\beta}_1(\kappa) - \beta_1)$ is dependent on the magnitude of $\sigma_\epsilon^2$, we investigated the use of second and fourth moments to estimate each of the error variances.

Using Second and Fourth Moments to Estimate the Error Variances $\sigma_\epsilon^2$ and $\sigma_\delta^2$

O’Driscoll and Ramirez (2011) suggested using the fourth moment slope estimator

$$\beta_1^{mt} = \sqrt{(s_{xzyy} - 3s_{xy}s_{yy})/(s_{xxyy} - 3s_{xxy}^2)}$$

described by Gillard and Iles (2005, Eqn. 26) to smooth out the jump discontinuity between $\beta_1^{per}$ and $\beta_1^{hor}$. For $s_{xy} > 0$, the authors define $\beta_1^{mt4}$ as:

$$\beta_1^{mt4} = \begin{cases} 
\beta_1^{gm} & \text{undefined} \\
\beta_1^{per} & \beta_1^{mt} < \beta_1^{per} \\
\beta_1^{mt} & \beta_1^{per} \leq \beta_1^{mt} \leq \beta_1^{hor} \\
\beta_1^{hor} & \beta_1^{mt} > \beta_1^{hor}
\end{cases}$$
The estimators $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ of Equation (2) are computed using $\hat{\beta}_1 = \beta_1^{mt4}$ and we constrain the moment estimators so that the error variances are non-negative. We then use these estimates in Madansky’s adjusted geometric mean estimator

$$\bar{\beta}_1^{gm} = \text{sgn}(s_{xy}) \sqrt{\frac{s_{yy} - \hat{\sigma}_2^2}{s_{xx} - \hat{\sigma}_1^2}}.$$  

Some authors such as Al-Nasser (2012) refer to Gillard and Iles (2009) who suggest using

$$\hat{\kappa}(\beta_1^{mt4}) = \frac{(s_{yy} - \beta_1^{mt4}s_{xy})}{(s_{xx} - s_{xy}/\beta_1^{mt4})}$$  

as an estimator for $\kappa$ in Equation (4) to find $\beta_1^{kap}$. However, as noted in O’Driscoll and Ramirez (2011, Proposition 3), there is a circular relationship between $\kappa$ and $\beta_1^{kap} = \bar{\beta}_1(\kappa)$ such that $\bar{\beta}_1(\kappa(\beta_1)) = \beta_1$; and in particular, $\bar{\beta}_1(\kappa(\beta_1^{mt4}) = \beta_1^{mt4}$.

In a second simulation study we again set $\beta_1 = 1$ and $\beta_0 = 0$. The $X$ data was generated from a uniform distribution on $(0, 20)$ so that the error variances $\{\sigma_1^2, \sigma_2^2\}$ varying over $\{1, 4, 9\}$ would have a strong impact on the bias of the regression slope. Table 2 records the expected values and standard deviations of our estimators for $\sigma_1^2$ and $\sigma_2^2$ using $\beta_1^{mt4}$ as an estimate for the slope $\beta_1$ in Equation (2).

<table>
<thead>
<tr>
<th>$\sigma_1^2$</th>
<th>$\sigma_2^2$</th>
<th>$E(\hat{\sigma}_1^2)$</th>
<th>$E(\text{std}(\hat{\sigma}_1^2))$</th>
<th>$E(\hat{\sigma}_2^2)$</th>
<th>$E(\text{std}(\hat{\sigma}_2^2))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.9542</td>
<td>0.6952</td>
<td>0.9485</td>
<td>0.6972</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1.2637</td>
<td>1.2084</td>
<td>3.4798</td>
<td>1.4718</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>1.7060</td>
<td>1.7808</td>
<td>7.7635</td>
<td>2.5742</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3.4872</td>
<td>1.4550</td>
<td>1.2518</td>
<td>1.2167</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3.7429</td>
<td>2.1852</td>
<td>3.7925</td>
<td>2.2096</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>4.0009</td>
<td>2.9326</td>
<td>8.2035</td>
<td>3.4808</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>7.7670</td>
<td>2.6010</td>
<td>1.7161</td>
<td>1.8138</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>8.1291</td>
<td>3.5325</td>
<td>4.0944</td>
<td>2.9942</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>8.3387</td>
<td>4.5750</td>
<td>8.4323</td>
<td>4.5883</td>
</tr>
</tbody>
</table>

Tables 3A, 3B and 3C record the expected means, standard deviations and mean square errors for the five estimators $\beta_1^{ver}, \beta_1^{hor}, \beta_1^{ver}, \beta_1^{gm}$ and $\beta_1^{gm}$ where $\beta_1^{mt4}$ is used in Equation (2) to estimate $\sigma_1^2$ and $\sigma_2^2$. Our values recorded in the set count are the number of times that $\beta_1^{mt4}$ was not defined, that $\sigma_2^2$ had to be increased to zero and that $\sigma_1^2$ had to be increased to zero respectively. Note
An Investigation of the Performance of Five Different Estimators in the Measurement Error Regression Model

that $\beta_1^{\text{hor}}$ requires the term under the radical to be non negative and hence for our simulation we chose the uniform distribution.

<table>
<thead>
<tr>
<th>Table 3A</th>
<th>${\beta_1 = 1, \beta_0 = 0, \mathcal{L}(X) = U(0, 20), n = 50, N = 5000}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\sigma_2^2 = 1, \sigma_1^2 = 1}$</td>
<td>${\sigma_2^2 = 4, \sigma_1^2 = 1}$</td>
</tr>
</tbody>
</table>
| $\begin{array}{|c|cc|cc|} 
 Estimator & Mean & St Dev & MSE & Mean & St Dev & MSE \\
$\beta_1^{\text{ver}}$ & 0.9718 & 0.03422 & 0.00196 & 0.8957 & 0.04776 & 0.01315 & 0.7907 & 0.05699 & 0.04617 \\
$\beta_1^{\text{hor}}$ & 1.0309 & 0.03664 & 0.00230 & 1.0329 & 0.05830 & 0.00448 & 1.0348 & 0.08248 & 0.00801 \\
$\beta_1^{\text{br}}$ & 1.0009 & 0.04129 & 0.00171 & 0.9592 & 0.05351 & 0.00453 & 0.8916 & 0.06864 & 0.01646 \\
$\beta_1^{\text{gm}}$ & 1.0009 & 0.03463 & 0.00112 & 0.9617 & 0.04995 & 0.00395 & 0.9040 & 0.06991 & 0.01293 \\
$\beta_1^{\text{agm}}$ & 1.0013 & 0.04129 & 0.00171 & 0.9934 & 0.06637 & 0.00445 & 0.9797 & 0.09378 & 0.00921 \\
\end{array}$

<table>
<thead>
<tr>
<th>Table 3B</th>
<th>${\beta_1 = 1, \beta_0 = 0, \mathcal{L}(X) = U(0, 20), n = 50, N = 5000}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\sigma_2^2 = 1, \sigma_1^2 = 4}$</td>
<td>${\sigma_2^2 = 4, \sigma_1^2 = 4}$</td>
</tr>
</tbody>
</table>
| $\begin{array}{|c|cc|cc|} 
 Estimator & Mean & St Dev & MSE & Mean & St Dev & MSE \\
$\beta_1^{\text{ver}}$ & 0.9724 & 0.05428 & 0.00371 & 0.8963 & 0.06238 & 0.01464 & 0.7912 & 0.08606 & 0.04831 \\
$\beta_1^{\text{hor}}$ & 1.1216 & 0.06808 & 0.01847 & 1.1235 & 0.07968 & 0.02160 & 1.1260 & 0.10460 & 0.02681 \\
$\beta_1^{\text{br}}$ & 1.0476 & 0.05854 & 0.00569 & 1.0038 & 0.07244 & 0.00526 & 0.9331 & 0.08596 & 0.01119 \\
$\beta_1^{\text{gm}}$ & 1.0442 & 0.05432 & 0.00490 & 1.0031 & 0.06651 & 0.00417 & 0.9428 & 0.07253 & 0.00853 \\
$\beta_1^{\text{agm}}$ & 1.0133 & 0.06869 & 0.00490 & 1.0049 & 0.09508 & 0.00696 & 0.9963 & 0.12581 & 0.01584 \\
\end{array}$

<table>
<thead>
<tr>
<th>Table 3C</th>
<th>${\beta_1 = 1, \beta_0 = 0, \mathcal{L}(X) = U(0, 20), n = 50, N = 5000}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\sigma_2^2 = 1, \sigma_1^2 = 9}$</td>
<td>${\sigma_2^2 = 4, \sigma_1^2 = 9}$</td>
</tr>
</tbody>
</table>
| $\begin{array}{|c|cc|cc|} 
 Estimator & Mean & St Dev & MSE & Mean & St Dev & MSE \\
$\beta_1^{\text{ver}}$ & 0.9730 & 0.07688 & 0.00664 & 0.9069 & 0.08138 & 0.01726 & 0.7917 & 0.08451 & 0.05053 \\
$\beta_1^{\text{hor}}$ & 1.2730 & 0.09325 & 0.08324 & 1.2751 & 0.11275 & 0.08837 & 1.2786 & 0.14135 & 0.09769 \\
$\beta_1^{\text{br}}$ & 1.1295 & 0.08763 & 0.02445 & 1.0825 & 0.10024 & 0.01685 & 1.0067 & 0.11329 & 0.01288 \\
$\beta_1^{\text{gm}}$ & 1.1123 & 0.07571 & 0.01834 & 1.0683 & 0.08281 & 0.01152 & 1.0041 & 0.08809 & 0.00778 \\
$\beta_1^{\text{agm}}$ & 1.0307 & 0.10038 & 0.01102 & 1.0185 & 0.13041 & 0.01735 & 1.0127 & 0.16925 & 0.02880 \\
\end{array}$

From Table 3A with $\sigma_2^2 = 9$ and $\sigma_1^2 = 1$ the MSEs for $\beta_1^{\text{ver}}$ and $\beta_1^{\text{hor}}$ are 0.04617 and 0.00801 respectively. The MSE for $\beta_1^{\text{agm}}$ is comparable to $\beta_1^{\text{hor}}$ with value 0.00921 and is smaller than the MSEs for the other three slope estimators. From Table 3C with $\sigma_2^2 = 1$ and $\sigma_1^2 = 9$ the MSEs for $\beta_1^{\text{ver}}$ and $\beta_1^{\text{hor}}$ are 0.00664 and 0.08324. The MSE for $\beta_1^{\text{agm}}$ is comparable to $\beta_1^{\text{ver}}$ with value 0.01102 and is again smaller then the the MSEs for the other three slope estimators. Thus $\beta_1^{\text{agm}}$ is robust when compared to a random choice between $\beta_1^{\text{ver}}$ and $\beta_1^{\text{hor}}$. 

153
Applications

Example 1

Riggs (1978) presented a small example with \( n = 6 \), \( X = [-4, -3, -2, 0, 3, 6] \) and \( Y = [-5, -1, -1, -2, 3, 6] \) and assumes that \( \sigma_y^2 = 4.933 \) is known. In the case where \( \sigma_y^2 \) is known Madansky proposed the estimator \( \hat{\beta}_{1}^{M_1} = (s_{yy} - \hat{\sigma}_y^2) / s_{xy} \) which for this data set gave an unsuitable line fit with negative slope. However, in practice suitable constraints have to be placed on the estimator equations such that \( 0 < \hat{\sigma}_x^2 < s_{xx} \) and \( 0 < \hat{\sigma}_x^2 < s_{yy} \). Our estimated value for \( \sigma_x^2 \) is 1.55 which yields \( \hat{\beta}_{1}^{M_1} = 0.946 \). In the case where \( \sigma_y^2 \) is known Madansky proposed the estimator \( \hat{\beta}_{1}^{M_2} = s_{xy} / (s_{yy} - \sigma_y^2) \) which yields \( \hat{\beta}_{1}^{M_2} = 1.063 \) with an estimated value for \( \sigma_y^2 = .082 \). (see Riggs(1978, Eqns. 35 and 37, p. 1320)). With these estimates of \( \sigma_x^2 \) and \( \sigma_y^2 \) Madansky’s adjusted geometric mean estimator yields \( \hat{\beta}_{1}^{agm} = 0.9522 \). Thus the anomaly with the original Madansky estimate does not exist with Madansky’s adjusted geometric mean estimator.

Example 2

Mamun et al. (2013) introduced a nonparametric estimator to the slope for measurement error models. They considered the serum kanamycin data set from Kelly (1984) with \( n = 20 \) for which they found the estimator of the slope \( \hat{\beta}_1 = 1.0 \) with \( \hat{\sigma}_x^2 = 3.4 \) and \( \hat{\sigma}_y^2 = 5.4 \). Her assumption of \( k = 1 \) is equivalent to \( \hat{\beta}_1 = \beta_1^{per} \), the perpendicular estimator of Adcock (1878). We compute the Jackknife estimator for the slope \( \hat{\beta}_1^{agm} \) and intercept \( \hat{\beta}_0^{agm} \) and their respective standard deviations and use this estimate to find variance estimators for \( \sigma_x^2 \) and \( \sigma_y^2 \). These values are shown in Column 4 in Table 4. The values in Column 5 are our Jackknife estimators for the perpendicular estimator used by Kelly (1984). Our methodology has as its only restriction that the error variances must be positive. The results are summarised in Table 4.

<table>
<thead>
<tr>
<th>( \hat{\beta}_0 )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\sigma}_x^2 )</th>
<th>( \hat{\sigma}_y^2 )</th>
<th>( \kappa = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mamun et al.</td>
<td>-1.16 (4.97)</td>
<td>-0.11 (0.7)</td>
<td>-4.69 (1.01)</td>
<td>-1.23 (1.16)</td>
</tr>
<tr>
<td>agm</td>
<td>1.07 (0.26)</td>
<td>1.0 (NA)</td>
<td>1.24 (0.05)</td>
<td>1.07 (0.06)</td>
</tr>
</tbody>
</table>

Table 4: Serum Kanamycin Data
An Investigation of the Performance of Five Different Estimators in the Measurement Error Regression Model

From the data of this particular sample, the adjusted geometric mean estimator for $\beta_1$ gives the linear calibration between the serum kanamycin levels in blood samples drawn simultaneously from an umbilical catheter and a heel venipuncture and is shown in the graph.

Conclusion

It has been demonstrated that the bias of the second moment estimator for the regression slope in the measurement error model is dependent on the magnitude of $\bar{\kappa} \sigma_1^2 - \sigma_\delta^2$. Our simulation studies suggest that using second and fourth moments to estimate each of the error variances in Equation (9) yields improvement in bias and mean square error over the ordinary least squares estimators $\beta_{1\text{er}}^\text{er}$ and $\beta_{1\text{sr}}^\text{hor}$, the geometric mean estimator, $\beta_{1\text{m}}^\text{gm}$, and the perpendicular estimator $\beta_{1\text{p}}^\text{per}$.

References


