

Introduction

Group theory is one of the most important areas of mathematics, with many applications throughout mathematics and science. A crowning achievement of group theory (indeed, of all mathematics) in the twentieth century was the classification of all finite simple groups. The results of this project will be mined for decades, if not longer, and its implications cannot be overestimated. Key to the advancement of group theory has been the discipline of representation theory, the study of the ways in which one mathematical object may “act” on another. Studying an object’s representations often yields much important information about that object’s structure.

The finite groups $G(q)$ of Lie type make up the bulk of all finite simple groups, so attention naturally focuses in this direction. These groups are defined as the fixed-point sets of algebraic groups under specified group automorphisms. For example, the finite group $SL_n(q) = SL_n(\mathbb{F}_q)$ for $q = p^r$ is the finite subgroup of $SL_n(\overline{\mathbb{F}}_p)$ that is fixed under the group automorphism raising each matrix entry to the p^r -th power. From this definition it is clear that the study of algebraic groups is important to the study of group theory.

I am interested in the representation theory and cohomology of finite groups, algebraic groups, Lie algebras, quantized enveloping algebras, q -Schur algebras, and other related mathematical objects. I am especially interested in connections between these different mathematical objects, and in ways that the structure of one informs the structure of another. My thesis work explores connections between the cohomology of Frobenius kernels of algebraic groups and the cohomology of Frobenius–Lusztig kernels of quantized enveloping algebras. My long-term research goals include exploiting already-known connections between quantized enveloping algebras, q -Schur algebras and finite groups of Lie type to obtain new results on the cohomology of finite general linear groups.

Thesis Work

Quantized enveloping algebras were introduced independently by Drinfel’d [7] and Jimbo [15] around 1985 as a tool to find solutions to the quantum Yang–Baxter equation, an equation arising in the field of statistical mechanics. The quantum algebra $\mathbb{U}_q(\mathfrak{g})$ with parameter q (an indeterminate) associated to a simple complex Lie algebra \mathfrak{g} is a $\mathbb{C}(q)$ -algebra defined, loosely speaking, as a certain “ q -deformation” of the universal enveloping algebra of \mathfrak{g} . (Note: In the context of quantum algebras, q stands for an indeterminate, but in the context of finite groups of Lie type, q stands for a prime power.) Following Lusztig [17], one can specialize the parameter q to an arbitrary unit ζ in a field k by extending scalars from a $\mathbb{Z}[q, q^{-1}]$ -form $U_q(\mathfrak{g})$ in $\mathbb{U}_q(\mathfrak{g})$. The resulting algebra is denoted $U_\zeta(\mathfrak{g})$.

If $\zeta \in k$ is chosen to be a root of unity, then $U_\zeta(\mathfrak{g})$ contains a finite-dimensional subalgebra $u_\zeta(\mathfrak{g})$, called the Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$. The algebra $u_\zeta(\mathfrak{g})$ plays a role in the representation theory of $U_\zeta(\mathfrak{g})$ analogous to that played by the first Frobenius kernel G_1 of G in the rational representation theory of the corresponding algebraic group G over k [16]. In my thesis [8], I study the algebra $u_\zeta(\mathfrak{g})$ in the so-called quantum mixed case, that is, when $p := \text{char}(k)$ is odd and $\zeta \in k$ is a primitive ℓ -th root of unity. I show that, in exact analogy to the classical work Friedlander and Parshall [11] and of Andersen and Jantzen [1] for algebraic groups, and in agreement with the characteristic zero results of Ginzburg and Kumar [13] and of Bendel, Nakano, Parshall and Pillen [2], the cohomology ring $H^\bullet(u_\zeta(\mathfrak{g}), k)$ of $u_\zeta(\mathfrak{g})$ is in most cases isomorphic to $k[\mathcal{N}]$, the coordinate ring of the variety of nilpotent elements in \mathfrak{g} .

I also study the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$ ($r \geq 1$). These algebras are finite-dimensional subalgebras of $U_\zeta(\mathfrak{g})$ that exist only if $p > 0$. They correspond to the higher Frobenius kernels G_r of G . I obtain quantum versions of classical results of Friedlander and Parshall [10] for the algebra $U_\zeta(G_r)$ and its Borel subalgebra $U_\zeta(B_r)$. I show, for all $r \geq 1$, that the cohomology ring of $U_\zeta(B_r)$ is finitely-generated. Then, using the results I obtained on the cohomology of $u_\zeta(\mathfrak{g})$, and assuming mild restrictions on the parameters p and ℓ , I show that the cohomology ring of $U_\zeta(G_1)$ is finitely-generated whenever G is of Lie type A or D . I believe that this finite-generation result should hold for all Lie types, but cannot prove it at present.

Research Goals

My first research goal is centered on studying the representations and cohomology of the Frobenius–Lusztig kernels $u_\zeta(\mathfrak{g})$ for $p = \text{char}(k)$ arbitrary, but with especial emphasis on the quantum mixed case. I propose systematically studying the cohomology groups $\text{Ext}_{u_\zeta(\mathfrak{g})}^\bullet(L, M)$ for finite-dimensional $u_\zeta(\mathfrak{g})$ -modules L and M , beginning with $L = k$ the trivial $u_\zeta(\mathfrak{g})$ -module, and M either an irreducible $u_\zeta(\mathfrak{g})$ -module or a “standard” module for the ambient quantized enveloping algebra $U_\zeta(\mathfrak{g})$. Bezrukavnikov [3] has computed $\text{Ext}_{u_\zeta(\mathfrak{g})}^\bullet(k, M)$ for $k = \mathbb{C}$ and M a tilting module, but in general little is known about the structure of these Ext groups. Still, I believe the calculations should be eminently doable. The problem could be attacked through computer calculations with the computer algebra systems GAP and Magma (at least for \mathfrak{g} of small rank and for small values of p and ℓ). I also propose investigating connections between quantum algebras, Ringel–Hall algebras, and q -Schur algebras [6, Parts 4–5] in order to extract additional cohomological data. And the theory of canonical bases has already been used to great effect by Ryom–Hansen [18] to study the cohomology of the full quantum algebra $U_\zeta(\mathfrak{g})$; I would like to investigate the applications of canonical bases to the computation of cohomology for $u_\zeta(\mathfrak{g})$. Computing the structure of $\text{Ext}_{u_\zeta(\mathfrak{g})}^\bullet(k, M)$ as a module for the ring $H^\bullet(u_\zeta(\mathfrak{g}), k)$ would be an important first step toward determining the structure of the support variety for $u_\zeta(\mathfrak{g})$ determined by M . (Support varieties are geometric objects defined in terms of representation-theoretic data. Their use permits one to apply results from algebraic geometry to the study of representation theory.)

My second research goal is centered on studying the representations and cohomology of the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$. Quantized enveloping algebras have received little specific attention in the quantum mixed case, and the algebras $U_\zeta(G_r)$, which only exist if $p > 0$, have received almost no attention at all. I believe that they are objects ripe for study, with potentially fruitful connections to the higher Frobenius kernels of algebraic groups (objects which are, conversely, the subjects of much interest and study). My primary goal is to investigate the structure of the cohomology ring $H^\bullet(U_\zeta(G_r), k)$, and to determine whether it enjoys the same cohomological finiteness properties investigated for finite group schemes by Friedlander and Suslin [12] and for arbitrary reductive group schemes by Touzé and van der Kallen [19]. This goal is motivated by the desire to develop a theory of support varieties for the algebras $U_\zeta(G_r)$ and also by deep questions from the theory of finite-dimensional Hopf algebras. For example, is the cohomology ring of a finite-dimensional Hopf algebra always finitely-generated? I conjecture that, at least for the higher Frobenius–Lusztig kernels, the answer is yes.

My third research goal is to study the modular representation theory of q -Schur algebras, little q -Schur algebras, and finite general linear groups. The modular representation theory of a finite group $G(q)$ of Lie type splits into two classes: the defining-characteristic theory concerns modules for $G(q)$ over fields having the same characteristic p as that defining $G(q)$ (i.e., $p \mid q$); the non-defining or cross-characteristic theory concerns modules for $G(q)$ over fields of characteristic different from that defining $G(q)$ (i.e., $p \nmid q$). My goal is to use already-established results relating the cross-characteristic representation theory of $GL_n(q)$ to the modular representation theory of q -Schur algebras [4], as well as results relating q -Schur algebras to $U_\zeta(\mathfrak{gl}_n)$ [9], in order to translate new results on the modular representation theory of $U_\zeta(\mathfrak{gl}_n)$ into new results on the cross-characteristic representation theory of $GL_n(q)$. This work complements and provides motivation for the first two research goals. Potential applications are virtually unlimited. For example, Robert Guralnick [14] has conjectured that if G is an arbitrary finite group, and if V is a finite-dimensional faithful absolutely irreducible module for G , then $\dim H^1(G, V) \leq m$ for some fixed $m \in \mathbb{N}$. Cline, Parshall and Scott [5] have discovered evidence in favor of this conjecture when $G = G(q)$ is a finite group of Lie type and V is a defining-characteristic representation of G . Any new information on the cross-characteristic representation theory of $GL_n(q)$ could help to extend this result to non-defining characteristic representations as well.

References

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