Spectral Sequences

Theorem (McCleary, *A User's Guide to Spectral Sequences*, Theorem 2.6)**.** Each filtered differential graded module (A, d, F^*) determines a spectral sequence, $\{E_r^{*,*}, d_r\}$, $(r = 1, 2, \ldots)$ with d_r of bidegree $(r, 1-r)$ and $E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A)$. Suppose further that the filtration is bounded, that is, for each dimension n, there are values $s = s(n)$ and $t = t(n)$, so that $\{0\} \subset F^sA^n \subset F^{s-1}A^n \subset \cdots \subset F^{t+1}A^n \subset F^tA^n = A^n$. Then the spectral sequence converges to $H(A, d)$, that is,

$$E_\infty^{p,q} \cong F^pH^{p+q}(A, d)/F^{p+1}H^{p+q}(A, d)$$

In constructing the spectral sequence, one makes the following definitions:

$$\begin{aligned}
Z_r^{p,q} &= F^pA^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \\
B_r^{p,q} &= F^pA^{p+q} \cap d(F^{p-r}A^{p+q-1}) \\
Z_\infty^{p,q} &= \ker d \cap F^pA^{p+q} \\
B_\infty^{p,q} &= \operatorname{im} d \cap F^pA^{p+q} \\
E_r^{p,q} &= Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q})
\end{aligned}$$

The differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is induced by d. The assumption on the boundedness of the filtration implies that $Z_r^{p,q} = Z_\infty^{p,q}$ and $B_r^{p,q} = B_\infty^{p,q}$ for $r > s(p+q+1) - p$ and $r \geq p - t(p+q-1)$.

[Weibel, *An introduction to homological algebra*, Theorem 5.4.1] constructs the corresponding spectral sequence in the homological case where d has degree -1 and F is an increasing filtration (i.e., $F^sA \subset F^{s-1}A$). Weibel proves the convergence result for bounded filtrations in Theorem 5.5.1.

Proposition (McCleary, Proposition 2.11)**.** For a filtered differential graded R-module (A, d, F), the spectral sequence associated to the (decreasing) filtration and the spectral sequence associated to the exact couple are the same.

Given a double complex $\{M^{*,*}, d', d''\}$ with d' of bidegree $(1, 0)$ and d'' of bidegree $(0, 1)$, let $H_I^{*,*}(M) = H(M^{*,*}, d')$, and let $H_{II}^{*,*}(M) = H(M^{*,*}, d'')$. We have two filtrations of $M^{*,*}$: The column-wise filtration is given by $F_I^p(\mathrm{Tot}(M)) = \oplus_{r \geq p} M^{r,s}$. The row-wise filtration is given by $F_{II}^p(\mathrm{Tot}(M)) = \oplus_{s \geq p} M^{r,s}$.

Theorem (McCleary, Theorem 2.15)**.** Given a double complex $\{M^{*,*}, d', d''\}$ there are two spectral sequences, $\{{}_IE_r^{*,*}, {}_Id_r\}$ and $\{{}_{II}E_r^{*,*}, {}_{II}d_r\}$ with

$${}_IE_2^{*,*} \cong H_I^{*,*}H_{II}(M) \qquad \text{and} \qquad {}_{II}E_2^{*,*} \cong H_{II}^{*,*}H_I(M)$$

If $M^{p,q} = \{0\}$ when $p < 0$ or $q < 0$, then both spectral sequences converge to $H^*(\mathrm{Tot}(M), d)$.

The main tools used in proving this theorem are [McCleary, Theorem 2.6 and Proposition 2.11]. One notes that the column-wise and row-wise filtrations of $M^{*,*}$ are decreasing filtrations, that the total differential $d = d' + d''$ respects each filtration, and the filtrations are bounded because $M^{*,*}$ is a first quadrant double complex. One applies Theorem 2.6 to obtain descriptions of the E_1 and to obtain the convergence conclusion. One applies Proposition 2.11 to derive the expressions for the E_2 terms.

Weibel proves this result under a homological setup in Section 5.6.

Theorem (Hilton and Stammbach, *A Course in Homological Algebra*, Theorem 9.3)**.** (*Grothendieck spectral sequence*) Given $F : \mathfrak{U} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{C}$, assume that if I is an injective object in $\mathfrak{U}$, then $F(I)$ is G-acyclic. Then there is a spectral sequence $\{E_n(A)\}$ corresponding to each object A of $\mathfrak{U}$ such that

$$E_2^{p,q} = (R^pG)(R^qF)(A) \Rightarrow R^{p+q}(GF)(A)$$

which converges finitely to the graded object associated with $\{R^{p+q}(GF)(A)\}$ suitably filtered.

Proof. Take an injective resolution of A in $\mathfrak{U}$, $\mathbf{I} : I_0 \to I_1 \to I_2 \to \cdots$. Apply F to obtain the cochain complex in $\mathfrak{B}$, $FI_0 \to FI_1 \to I_2 \to \cdots$. We write F_r for FI_r, and display the cocycles and coboundaries of this chain complex as

$$Z_0 \hookrightarrow F_0 \twoheadrightarrow B_1 \hookrightarrow Z_1 \hookrightarrow F_1 \twoheadrightarrow B_2 \hookrightarrow Z_2 \hookrightarrow F_2 \twoheadrightarrow \cdots$$

We resolve this complex as

$$\begin{array}{ccccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K_{0,1} & \hookrightarrow & J_{0,1} & \twoheadrightarrow & L_{1,1} & \hookrightarrow & K_{1,1} & \hookrightarrow & J_{1,1} & \twoheadrightarrow & L_{2,1} & \hookrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K_{0,0} & \hookrightarrow & J_{0,0} & \twoheadrightarrow & L_{1,0} & \hookrightarrow & K_{1,0} & \hookrightarrow & J_{1,0} & \twoheadrightarrow & L_{2,0} & \hookrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
Z_0 & \hookrightarrow & F_0 & \twoheadrightarrow & B_1 & \hookrightarrow & Z_1 & \hookrightarrow & F_1 & \twoheadrightarrow & B_2 & \hookrightarrow & \cdots
\end{array} \tag{1}$$

where each column is an (augmented) injective resolution of the object at its base, and $K_{r,s} \hookrightarrow J_{r,s} \twoheadrightarrow L_{r+1,s}$ is exact. We accomplish this as follows: We resolve $Z_0 \hookrightarrow F_0 \twoheadrightarrow B_1$. Given the resolution of B_1, we choose an injective resolution of Z_1/B_1 and resolve $B_1 \hookrightarrow Z_1 \twoheadrightarrow Z_1/B_1$. Then use an arbitrary resolution of B_2 to resolve $Z_1 \hookrightarrow F_1 \twoheadrightarrow B_2$. Proceeding in this fashion, we construct the resolution. Note that by construction, $Z_r/B_r \hookrightarrow K_{r,0}/L_{r,0} \to K_{r,1}/L_{r,1} \to K_{r,2}/L_{r,2} \to \cdots$ is an injective resolution of Z_r/B_r. We now have a resolution of $FI_0 \to FI_1 \to I_2 \to \cdots$,

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
J_{0,1} & \to & J_{1,1} & \to & J_{2,1} & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
J_{0,0} & \to & J_{1,0} & \to & J_{2,0} & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
FI_0 & \to & FI_1 & \to & FI_2 & \to & \cdots
\end{array}$$

such that each row is a cochain complex and the r-th column is an (augmented) injective resolution of FI_r $(r = 0, 1, 2, \ldots)$. Call the double complex obtained by deleting the bottom row M. Apply G

to M to obtain the double complex GM:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \\
GJ_{0,2} & \xrightarrow{d'} & GJ_{1,2} & \xrightarrow{d'} & GJ_{2,2} & \xrightarrow{d'} & \cdots \\
\uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \\
GJ_{0,1} & \xrightarrow{d'} & GJ_{1,1} & \xrightarrow{d'} & GJ_{2,1} & \xrightarrow{d'} & \cdots \\
\uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \\
GJ_{0,0} & \xrightarrow{d'} & GJ_{1,0} & \xrightarrow{d'} & GJ_{2,0} & \xrightarrow{d'} & \cdots
\end{array}
$$

We now apply [McCleary, Theorem 2.15]. Considering the column-wise filtration F_I, we get

$$
{}_IE_1^{p,q} = H^q(GJ_{p,*}, d'') = R^qGI_p = \begin{cases} GFI_p & q = 0 \\ 0 & q \neq 0 \end{cases}
$$

$$
{}_IE_2^{p,q} = H^p\left(H^q(GJ, d''), d'\right) = \begin{cases} R^p(GF)(A) & q = 0 \\ 0 & q \neq 0 \end{cases}
$$

The ${}_IE_2$ terms all live in the x-axis. We conclude that ${}_IE_2^{p,q} = {}_IE_\infty^{p,q}$ by considering the degrees of the differentials d_r ($r \geq 2$). Moreover, since there is only one nonzero ${}_IE_\infty^{p,q}$ term in each column, we must also have $H^p(\mathrm{Tot}(M)) = R^p(GF)(A)$ because the filtration on $H^*(\mathrm{Tot}(M))$ is finite (as a consequence of the filtration on $\mathrm{Tot}(M)$ being finite).

Now consider the row-wise filtration F_II. Since all of the objects in the resolution (1) are injective, all of the monomorphisms split. Thus when we apply the functor G we maintain all exactness relations. In particular, $G(K_{r,s}/L_{r,s}) = GK_{r,s}/GL_{r,s}$. Also note that $Z_r/B_r = R^rF(A)$. Now,

$$
{}_{II}E_1^{p,q} = H^q(GJ_{*,p}, d') = GK_{q,p}/GL_{q,p} = G(K_{q,p}/L_{q,p})
$$

Using the facts that $Z_r/B_r \hookrightarrow K_{r,0}/L_{r,0} \to K_{r,1}/L_{r,1} \to K_{r,2}/L_{r,2} \to \cdots$ is an injective resolution of Z_r/B_r, and that $Z_r/B_r = R^rF(A)$, we get

$$
\begin{aligned}
{}_{II}E_2^{p,q} &= \left(\ker d'' : G(K_{q,p}/L_{q,p}) \to G(K_{q,p+1}/L_{q,p+1})\right) / \left(\operatorname{im} d'' : G(K_{q,p-1}/L_{q,p-1}) \to G(K_{q,p}/L_{q,p})\right) \\
&= (R^pG)(Z_q/B_q) \\
&= (R^pG)(R^qF)(A)
\end{aligned}
$$

This proves the theorem. □

Theorem (Hilton and Stammbach, Theorem 9.5)**.** (Lyndon-Hochschild-Serre) Given the short exact sequence of groups $N \hookrightarrow G \twoheadrightarrow G/N$ and a G-module A, there is a spectral sequence $\{E_n(A)\}$ such that

$$
E_2^{p,q} = H^p\left(G/N, H^q(N, A)\right) \Rightarrow H^{p+q}(G, A)
$$

which converges finitely to the graded group associated with $\{H^q(G, A)\}$.

Proof. Let $\mathfrak{U} = G - Mod$, $\mathfrak{B} = G/N - Mod$, and let $\mathfrak{B} = \mathbf{Ab}$. Let $F : \mathfrak{U} \to \mathfrak{B}$ be defined by $F(A) = \mathrm{Hom}_N(\mathbb{Z}, A) = A^N$, and let $G : \mathfrak{B} \to \mathfrak{C}$ be defined by $G(B) = \mathrm{Hom}_{G/N}(\mathbb{Z}, B) = B^{G/N}$. The result now follows from the Grothendieck spectral sequence. □