Simple Modules for Connected Reductive Algebraic Groups

It is my intention to describe certain general properties of simple (rational) G-modules when G is a connected reductive algebraic group. This exposition is based in large part on Jantzen's *Representations of Algebraic Groups* (Chapter 2, Part II), and Humphreys' *Linear Algebraic Groups.*

For simplicity, we assume that k is an algebraically closed field.

Preliminaries

Let G be a connected reductive affine algebraic group over k. Let T be a maximal torus of G with character group $X(T)$. Let B be a Borel subgroup of G containing T, and let $U = R_u(B)$ the unipotent radical of B. We have $B \cong U \rtimes T$. Let B^+ denote the opposite Borel subgroup, and let $U^+ = R_u(B^+)$. The subgroups T, B, U are characterized as being maximal connected diagonalizable, solvable, and unipotent subgroups of G, respectively.

Let $\mathcal{G} = \operatorname{Lie} G$ the Lie algebra of G, let $\Phi \subset X(T)$ denote the roots of $\mathcal{G}$ relative to T, and let Φ^+ denote the positive roots relative to B^+. Then

$$\mathcal{G} = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in \Phi} (\operatorname{Lie} G)_\alpha$$

with $(\operatorname{Lie} G)_\alpha = \operatorname{Lie} U_\alpha$, U_α the unique connected T-stable subgroup of G having Lie algebra $(\operatorname{Lie} G)_\alpha$. Associated to the roots $\Phi \subset X(T)$, we have the collection of coroots $\Phi^\vee \subset Y(T)$.

Representations of Connected Reductive Algebraic Groups

Let M be a rational G-module. Then M is a direct sum of weight spaces for the action of T:

$$M = \bigoplus_{\lambda \in X(T)} M_\lambda$$

where $M_\lambda = \{m \in M : t.m = \lambda(t)m, \ \forall\, t \in T\}$. The functor $M \mapsto M_\lambda$ is exact for each weight λ of T on M.

From the fact that every G-module is locally finite we deduce that all simple G-modules are finite dimensional. Recall that any unipotent algebraic group acting on a finite dimensional vector space has a fixed point (i.e., an eigenvector with eigenvalue one). It follows that if M is a nonzero G-module, then $M^{U^+} \neq 0$ (where here M^{U^+} denotes the space of fixed points for the action of U^+ on M). Since T normalizes U^+, M^{U^+} is a T-submodule of M and hence a direct sum of weight spaces. Similar remarks hold with U substituted for U^+.

Induction

Let H be a subgroup of G and let M be a nonzero H-module. Define actions of G and H on $M \otimes k[G]$ as follows. Let G act trivially on M and via the left regular representation on $k[G]$. Let H act as given on M and via the right regular representation on $k[G]$. These actions commute, so the subspace $(M \otimes k[G])^H$ of fixed points for the action of H is a G-submodule. We denote this submodule by $\operatorname{ind}_H^G M$ and call it the induced module of M from H to G.

If we view G and M as affine group schemes over k (where $G(A) = \mathrm{Hom}_{k-alg}(k[G], A)$ and $M(A) = M \otimes_k A$ for all k-algebras A), then we may interpret $M \otimes k[G]$ functorially as $\mathrm{Mor}(G, M)$, the collection of natrual transformations from the k-functor G to the k-functor M. (This follows from Yoneda's Lemma.) Under this interpretation, we have

$$\mathrm{ind}_H^G M = \left\{ f \in \mathrm{Mor}(G, M) : f(gh) = h^{-1} f(g) \ \forall\ g \in G(A), h \in H(A), \text{ and } \forall\ k\text{-algebras } A \right\}$$

One can show that the functor ind_H^G is left exact and right adjoint to the restriction functor res_H^G. So $\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) \cong \mathrm{Hom}_H(\mathrm{res}_H^G N, M)$ for all G-modules N and H-modules M (Frobenius Reciprocity).

Simple Modules

For any $\lambda \in X(T)$, define a one-dimensional rational representation k_λ of B by $ut \mapsto \lambda(t) \in k$ ($u \in U$, $t \in T$). Note that every non-zero simple B-module M has this form, as by the Lie-Kolchin Theorem all irreducible modules for B are one-dimensional, $M^U \neq 0$ and M is a direct sum of weight spaces for T by the remarks above.

Denote the induced module $\mathrm{ind}_B^G k_\lambda$ by $H^0(\lambda)$.

Proposition. Let $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$.

(a) $\dim H^0(\lambda)^{U^+} = 1$ and $H^0(\lambda)^{U^+} = H^0(\lambda)_\lambda$.

(b) Each weight μ of $H^0(\lambda)$ satisfies $w_0\lambda \leq \mu \leq \lambda$.

(c) $L(\lambda) := \mathrm{soc}_G H^0(\lambda)$ is simple.

(d) $L(\lambda)^{U^+} = L(\lambda)_\lambda$ and $\dim L(\lambda)_\lambda = 1$. The weights μ of $L(\lambda)$ satisfy $w_0\lambda \leq \mu \leq \lambda$. The multiplicity of $L(\lambda)$ as a composition factor of $H^0(\lambda)$ is one.

(e) Any simple G-module is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X(T)$, $H^0(\lambda) \neq 0$.

We point out that (c) follows from (a), for if L_1, L_2 are distinct nonzero simple submodules of $H^0(\lambda)$, then $L_1 \oplus L_2 \subset \mathrm{soc}_G H^0(\lambda)$. But $0 \neq L_1^{U^+}$ and $0 \neq L_2^{U^+}$ by previous remarks, hence $\dim H^0(\lambda)^{U^+} \geq 2$, a contradiction. (d) also follows from (a) and the fact that $M \mapsto M_\lambda$ is an exact functor.

We sketch a proof of (e). Let V be a simple G-module, necessarily finite dimensional. Let λ be a maximal weight for the action of T on V. Then $\varphi \mapsto \varphi|_{V_\lambda}$ induces an isomorphism $\mathrm{Hom}_B(V, k_\lambda) \cong \mathrm{Hom}_k(V_\lambda, k) \neq 0$. Now by Frobenius Reciprocity, $0 \neq \mathrm{Hom}_B(V, k_\lambda) \cong \mathrm{Hom}_G(V, H^0(\lambda))$ for some $\lambda \in X(T)$. By the simplicity of V we conclude that V injects into $H^0(\lambda)$, and conclude from (c) that $V \cong L(\lambda)$. If also $V \cong L(\mu)$, we have by (d) that $L(\mu)_\mu \cong V^{U^+} \cong L(\lambda)_\lambda$ and these spaces are one dimensional, so we must have $\mu = \lambda$.

So every simple G-module has the form $L(\lambda)$ for some $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$. The answer to the question of which $\lambda \in X(T)$ can possibly occur is the same as for semisimple complex Lie algebras: $H^0(\lambda) \neq 0$ if and only if λ is dominant in the sense that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$. We denote the collection of all dominant weights by $X(T)_+$.

The implication $H^0(\lambda) \Rightarrow \lambda \in X(T)_+$ follows from the fact that the Weyl group permutes the root spaces of $H^0(\lambda)$: $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ is a weight of T on $H^0(\lambda)$ for all $\alpha \in \Phi$. But

λ is a maximal weight of $H^0(\lambda)$ by part (b) of the proposition, so we must have $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$. On the other hand, to show $\lambda \in X(T)_+ \Rightarrow H^0(\lambda) \neq 0$, given $\lambda \in X(T)_+$ one constructs some $f \in k[G]$ satisfying $f(gb) = \lambda(b)^{-1} f(g)$ for all $g \in G, b \in B$. (One begins by defining such a regular function on the dense open subset U^+B, and then showing that this function can be extended to all of G.)

Recall that the algebra of distributions $\mathrm{Dist}(G)$ is defined by

$$\mathrm{Dist}(G) = \left\{ \mu \in k[G]^* : \mu(I_1^{n+1}) = 0 \text{ for some } n \geq 1 \right\}$$

where $I_1 \lhd k[G]$ is the ideal defining the identity $1 \in G$. $\mathrm{Dist}(G)$ has a basis

$$\left\{ \prod_{\alpha<0} X_{\alpha,n(\alpha)} \prod_{i=1}^{r} H_{i,m(i)} \prod_{\alpha>0} X_{\alpha,n'(\alpha)} \,\middle|\, n(\alpha), m(i), n'(\alpha) \in \mathbb{N} \right\}$$

where $X_{\alpha,m} = (X_\alpha)^m/(m!)$, $H_{i,m} = \binom{H_i}{m}$, and products are taken over some arbitrary fixed ordering of the roots. (Here X_α is a basis vector for $(\mathrm{Lie}\, G)_\alpha$, and $H_1, \ldots, H_r$ is a basis for $\mathrm{Lie}\, T$.) We write $\mathrm{Dist}(U)$ for the subalgebra generated by all $(X_\alpha)_{\alpha<0}$, $\mathrm{Dist}(T)$ for the subalgebra generated by $H_1, \ldots, H_r$, and $\mathrm{Dist}(U^+)$ for the subalgebra generated by all $(X_\alpha)_{\alpha>0}$.

Let M be a G-module, $M' \subset M$ a subspace. Then M is naturally a $\mathrm{Dist}(G)$-module. Since $k[G]$ is noetherian (by the assumption that G is algebraic) and integral, we have that M' is a G-submodule of M if and only if it is a $\mathrm{Dist}(G)$-submodule.

Consider again the simple G-module $L(\lambda)$, $\lambda \in X(T)_+$. Let $0 \neq v \in L(\lambda)_\lambda$. By simplicity, $L(\lambda) = \mathrm{Dist}(G)v$. Using the fact that $X_\alpha M_\mu \subset M_{\mu+\alpha}$ for any G-module M, and the fact that the H_i act on the weight spaces of any G-module as scalar multiplications, we conclude from the description of the basis of $\mathrm{Dist}(G)$ that $L(\lambda) = \mathrm{Dist}(U)v$ because λ is a maximal weight of $L(\lambda)$.

Finally, we note the existence of certain G-modules corresponding to the Verma modules in the representation theory of semisimple complex Lie algebras. We take for granted that $\dim H^0(\lambda) < \infty$ for all $\lambda \in X(T)$. (This is of course trivial if $\lambda \notin X(T)_+$, for if $\lambda \notin X(T)_+$ then $H^0(\lambda) = 0$.)

For $\lambda \in X(T)_+$, define $V(\lambda) = H^0(-w_0\lambda)^*$. One can show that $V(\lambda) \cong {}^\tau H^0(\lambda)$, where ${}^\tau H^0(\lambda)$ denotes the G-module that is $H^0(\lambda)^*$ as a vector space, but with G-action twisted by the involution $\tau : G \to G$. $g\varphi = \varphi \circ \tau(g)$ for all $\varphi \in H^0(\lambda)^*$. ($\tau$ is an anti-automorphism of G of order two that restricts to the identity on T and maps $U_\alpha \mapsto U_{-\alpha}$ for all $\alpha \in \Phi$.)

Proposition. $V(\lambda)$ is generated by a B^+-stable line of weight λ, and any G-module generated by a B^+-stable line of weight λ is a homomorphic image of $V(\lambda)$.

Proof. First one establishes natural isomorphisms

$$\mathrm{Hom}_G(V(\lambda), M) \xrightarrow{\sim} \mathrm{Hom}_B(w_o\lambda, M) \xrightarrow{\sim} (M^U)_{w_0\lambda} \xrightarrow{\sim} (M^{U^+})_\lambda$$

If ϵ denotes the evaluation map $\epsilon : H^0(w_0\lambda) \to k_{w_0\lambda}$, $f \overset{\epsilon}{\mapsto} f(1)$, then the isomorphisms are given by $\varphi \mapsto \varphi \circ \epsilon^*$, $\varphi \mapsto \varphi(\epsilon^*(1))$, and $\varphi(\epsilon^*(1)) \mapsto \dot{w}_0\varphi(\epsilon^*(1)) = \varphi(\dot{w}_0\epsilon^*(1))$, respectively.

Set $v = \dot{w}_0\epsilon^*(1)$, a B^+-eigenvector of weight λ, and let M be the G-submodule of $V(\lambda)$ generated by v. Then the projection map $\varphi : V(\lambda) \to V(\lambda)/M$ satisfies $\varphi(v) = 0$. By the above isomorphisms we must have $\varphi = 0$, i.e., $M = V(\lambda)$.

Now let M be an arbitrary G-module generated by a B^+-eigenvector $m \in M$ of weight λ. We have a nonzero homomorphism of B^+-modules $k_\lambda \to M$ induced by the map $1 \mapsto m$. Also, the map $\varphi \mapsto \varphi(1)$ induces an isomorphism $\operatorname{Hom}_{B^+}(k_\lambda, M) \xrightarrow{\sim} (M^{U^+})_\lambda$. Combining this isomorphism with the previous string of natural ismorphisms, we get a homomorphism of G-modules $V(\lambda) \to M$ mapping $v \mapsto m$. Since m generates M by assumption, we conclude that M is a homomorphic image of $V(\lambda)$. □

Proposition. $V(\lambda)/\operatorname{rad}_G V(\lambda) \cong L(\lambda)$.

Proof. First recall that $L(\lambda) = \operatorname{soc}_G H^0(\lambda)$ is the largest semisimple submodule of $H^0(\lambda)$, and $V(\lambda)/\operatorname{rad}_G V(\lambda)$ is the largest semisimple quotient of $V(\lambda)$. The functor $M \mapsto {}^\tau M$ described above is exact. Moreover, $L(\lambda) \cong {}^\tau L(\lambda)$, because $M \mapsto {}^\tau M$ sends simple modules to simple modules, the weights of ${}^\tau M$ are the same as those of M, and any simple module is determined by its weights.

Applying the functor ${}^\tau(?)$ to the inclusion $L(\lambda) \hookrightarrow H^0(\lambda)$, we obtain the surjection $V(\lambda) \cong {}^\tau H^0(\lambda) \twoheadrightarrow {}^\tau L(\lambda) \cong L(\lambda)$. $L(\lambda)$ is then a semisimple quotient of $V(\lambda)$, and it must be the largest such quotient because applying the functor ${}^\tau(?)$ a second time would yield an inclusion of a semisimple G-module into $H^0(\lambda)$. (${}^{\tau\tau}M \cong M$) Thus $V(\lambda)/\operatorname{rad}_G V(\lambda) \cong L(\lambda)$. □

Given a dominant weight $\lambda \in X(T)_+$, to construct $L(\lambda)$ it suffices to construct a G-module containing a B^+-stable line kv^+ of weight λ. For then the G-module M generated by v^+ is a nonzero homomorphic image of $V(\lambda)$. By the above proposition, $V(\lambda)$ contains a unique maximal submodule $\operatorname{rad}_G V(\lambda)$ with irreducible quotient $L(\lambda)$. We conclude that some quotient of M is isomorphic to $L(\lambda)$.