Theorem. Let R be a commutative ring with identity having no nonzero nilpotent elements. Then $R[x]^* = R^*$.

Proof. The inclusion $R^* \subseteq R[x]^*$ is trivial. Let $p(x) \in R[x]^*$, and write $p(x) = \sum_{k=0}^{n} a_k x^k$. Suppose $\deg(p) > 0$. We have $a_0 \in R^*$, because a_0 is the homomorphic image of p under the evaluation homomorphism $p(x) \mapsto p(0)$, and the homomorphic image of a unit is a unit. Let $q(x) = p(x)^{-1}$, and write $q(x) = \sum_{j=0}^{m} b_j x^j$. If $\deg(q) = 0$, then $b_0 a_n = b_0 a_{n-1} = \cdots = b_0 a_1 = 0$, which implies that $b_0 \in R$ is a zero divisor because $a_n \neq 0$. This is a contradiction, because $q(x)$ was assumed to be a unit. So $\deg(q) > 0$. Now we must have $a_n b_m = 0$. Consider the coefficient of x^{n+m-1} in $p(x)q(x)$. We have

$$\begin{aligned}
0 &= a_{n-1}b_m + a_n b_{m-1} \Rightarrow \\
0 &= a_n(a_{n-1}b_m) + a_n(a_n b_{m-1}) \\
&= a_{n-1}(a_n b_m) + a_n^2 b_{m-1} \\
&= 0 + a_n^2 b_{m-1} = a_n^2 b_{m-1}
\end{aligned}$$

Now suppose $a_n^{j+1} b_{m-i} = 0$ for all $0 \leq i \leq j$. If $j+1 < m+n$, the coefficient of $x^{m+n-(j+1)}$ must be zero. Then

$$\begin{aligned}
0 &= a_n b_{m-(j+1)} + a_{n-1} b_{m-j} + \cdots + a_{n-(j+1)} b_m \Rightarrow \\
0 &= a_n^{j+1} \left(a_n b_{m-(j+1)} + a_{n-1} b_{m-j} + \cdots + a_{n-(j+1)} b_m \right) \\
&= a_n^{j+2} b_{m-(j+1)} + a_{n-1}(a_n^{j+1} b_{m-j}) + \cdots + a_{n-(j+1)}(a_n^{j+1} b_m) \\
&= a_n^{j+2} b_{m-(j+1)} + 0 = a_n^{j+2} b_{m-(j+1)}
\end{aligned}$$

Conclude that $a_n^{j+1} b_{m-i} = 0$ for $0 \leq i \leq j$ and $j+1 < n+m$. (As usual, we have adopted the convention that $a_k, b_k = 0$ for $k < 0$, $a_k = 0$ for $k > n$, and $b_k = 0$ for $k > m$.) Now consider the coefficient of x^n in the product $p(x)q(x)$. We have

$$\begin{aligned}
0 &= a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n \Rightarrow \\
0 &= a_n^{(m-1)+1} (a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n) \\
&= a_n^{m+1} b_0 + a_{n-1}(a_n^m b_1) + \cdots + a_0(a_n^m b_n) \\
&= a_n^{m+1} b_0 + 0 = a_n^{m+1} b_0
\end{aligned}$$

Then $a_n^{m+1} = 0$ and a_n is nilpotent (because $b_0 \in R^*$ and hence is not a zero divisor). $\square$