Adjoint actions on integral forms in quantized enveloping algebras.

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October 18, 2008

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Let k be an algebraically closed field of characteristic $p \neq 2$.
Assume $p \neq 3$ if Φ has type G_2.

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Let $U_q(\mathfrak{g}) \subset \mathbb{U}_q(\mathfrak{g})$ denote the Lusztig integral form over $k[q, q^{-1}]$.
Given $\zeta \in k^\times$, let $U_\zeta(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes_{k[q,q^{-1}]} k$.

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Assume $\zeta \in k$ is a primitive ℓ-th root of unity.
The elements $E_\alpha, F_\alpha, K_\alpha \in U_\zeta(\mathfrak{g})$ $(\alpha \in \Pi)$ generate a finite-dimensional subalgebra of $U_\zeta(\mathfrak{g})$, denoted $u_\zeta(\mathfrak{g})$. It satisfies $U_\zeta(\mathfrak{g})//u_\zeta(\mathfrak{g}) \cong \text{hy}(G)$.

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Theorem (Ginzburg-Kumar, 1993)

Assume $k = \mathbb{C}$. Suppose $\ell > h$.
Then $H^{odd}(u_\zeta(\mathfrak{g}), \mathbb{C}) = 0$ and $H^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$.

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Question: What is $H^\bullet(u_\zeta(\mathfrak{g}), k)$ when $\mathrm{char}(k) > 0$?

Strategy: Mimic the arguments from characteristic zero.
Compute $H^\bullet(u_\zeta(\mathfrak{b}), k)$, then apply induction to get $H^\bullet(u_\zeta(\mathfrak{g}), k)$.

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Tool in computing $H^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$: The De Concini-Kac quantum algebra.
$\mathcal{U}_q(\mathfrak{g}) = k[q, q^{-1}]$-subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ $(\alpha \in \Pi)$.

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- Define a multiplicative (multi-)filtration on $\mathbb{U}_q(\mathfrak{u})$.
  This induces a multiplicative filtrations on $\mathcal{U}_\zeta(\mathfrak{u})$ and $u_\zeta(\mathfrak{u})$.
- Compute the cohomology of $\operatorname{gr} \mathcal{U}_\zeta(\mathfrak{u})$.
- Use an inductive argument to compute $H^\bullet(\operatorname{gr} u_\zeta(\mathfrak{u}), k)$.
- Use a spectral sequence $H^\bullet(\operatorname{gr} u_\zeta(\mathfrak{u}), k) \Rightarrow H^\bullet(u_\zeta(\mathfrak{u}), k)$ to compute $H^\bullet(u_\zeta(\mathfrak{b}), k)$.

The filtration on $\mathbb{U}_q(\mathfrak{u})$:

The algebra $\mathbb{U}_q(\mathfrak{u})$ has a PBW-type basis $F^a = F_{\gamma_N}^{a_N} \cdots F_{\gamma_1}^{a_1}$.
Order $\mathbb{N}^N$ using the usual total lexicographic order.
Given $a = (a_N, \ldots, a_1) \in \mathbb{N}^N$, define $\mathbb{U}_q(\mathfrak{u})_a = \langle F^b : b \leq a \rangle$.

To prove that this defines a multiplicative filtration on $\mathbb{U}_q(\mathfrak{u})$, we need to prove a "commutativity" result between root vectors $F_{\gamma_i}, F_{\gamma_j}$.

First of two results observed with the help of GAP:

Lemma

Let $S \subset \mathbb{Z}[q, q^{-1}]$ *be the multiplicatively closed subset generated by*

$$\begin{array}{ll} \{1\} & \textit{Type ADE} \\ \{q - q^{-1}, q^2 - q^{-2}\} & \textit{Type BCF} \\ \{q - q^{-1}, q^2 - q^{-2}, q^3 - q^{-3}\} & \textit{Type G} \end{array}$$

Set $\mathcal{A} = S^{-1}\mathbb{Z}[q, q^{-1}]$. *Let* $1 \leq i < j \leq N$. *Then in* $\mathbb{U}_q(\mathfrak{u})$ *we have*

$$F_{\gamma_i} F_{\gamma_j} = q^{(\gamma_i, \gamma_j)} F_{\gamma_j} F_{\gamma_i} + (*),$$

where $(*)$ *is an* $\mathcal{A}$*-linear combination of* F^m *with* $m_s = 0$ *unless* $i < s < j$.

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Observation: Contrary to some published versions of this lemma, we need the extra denominators whenever Φ has two root lengths.

Example

Let Φ be of type B_2 and write $\Pi = \{\alpha, \beta\}$ with α long.
Write $w_0 = s_\alpha s_\beta s_\alpha s_\beta$, so that

$$\Phi^+ = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{\alpha, \alpha+\beta, \alpha+2\beta, \beta\}.$$

According to the QuaGroup package of GAP, we have

$$F_{\gamma_1} F_{\gamma_3} = F_{\gamma_3} F_{\gamma_1} + (1 - q^{-2}) F_{\gamma_2}^{(2)}.$$

Since $\gamma_2 = s_\alpha(\beta)$, we have $F_{\gamma_2}^{(2)} = F_{\gamma_2}^2 / [2]_\beta^!$. Since β is short,

$$\frac{(1-q^{-2})}{[2]_\beta^!} = \frac{(1-q^{-2})}{q+q^{-1}} = \frac{(q^2-1)}{q(q^2+1)} \notin k[q, q^{-1}].$$

Now a result on adjoint actions:

Recall: A Hopf algebra H acts on itself via the adjoint action.
Let Δ and S denote the comultiplication and antipode in H.
If $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, then $\mathrm{Ad}(h)(u) = \sum h_{(1)} u S(h_{(2)})$.

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Let $\zeta \in k^\times$ be a primitive ℓ-th root of unity.

Let $\mathcal{B} = k[q, q^{-1}]_{(q-\zeta)}$.
Set $\mathcal{U}_q(\mathfrak{g})^{\mathcal{B}} = \mathcal{U}_q(\mathfrak{g}) \otimes_A \mathcal{B}$, $U_q(\mathfrak{g})^{\mathcal{B}} = U_q(\mathfrak{g}) \otimes_A \mathcal{B}$.
We have an inclusion of $\mathcal{B}$-forms, $\mathcal{U}_q(\mathfrak{g})^{\mathcal{B}} \subset U_q(\mathfrak{g})^{\mathcal{B}}$.

We want to show that there is an adjoint action of $U_q(\mathfrak{g})^{\mathcal{B}}$ on $\mathcal{U}_q(\mathfrak{g})^{\mathcal{B}}$, hence an adjoint action of $U_\zeta(\mathfrak{g})$ on $\mathcal{U}_\zeta(\mathfrak{g})$.

Recall $p = \mathrm{char}(k) \neq 2$. The algebra $U_q(\mathfrak{g})$ is generated by

$$\left\{ E_\alpha, F_\alpha, E_\alpha^{(p^i \ell)}, F_\alpha^{(p^i \ell)}, K_\alpha^{\pm 1} : \alpha \in \Pi, i \geq 0 \right\}$$

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Lemma (Arkhipov-Bezrukavnikov-Ginzburg, 2004)

$\mathrm{Ad}(E_\alpha^{(\ell)})(\mathcal{U}_q(\mathfrak{g})^{\mathcal{B}}) \subseteq \mathcal{U}_q(\mathfrak{g})^{\mathcal{B}}$ *and* $\mathrm{Ad}(F_\alpha^{(\ell)})(\mathcal{U}_q(\mathfrak{g})^{\mathcal{B}}) \subseteq \mathcal{U}_q(\mathfrak{g})^{\mathcal{B}}$.

Proof.

For $u \in \mathcal{U}_q(\mathfrak{g})$, we have $\mathrm{Ad}(E_\alpha^\ell)(u) \in (q - \zeta) \cdot \mathcal{U}_q(\mathfrak{g})$ and $\mathrm{Ad}(F_\alpha^\ell)(u) \in (q - \zeta) \cdot \mathcal{U}_q(\mathfrak{g})$. Up to a unit in $\mathcal{B}$, $E_\alpha^{(\ell)}$ is $E_\alpha^\ell/(q - \zeta)$, and similarly for $F_\alpha^{(\ell)}$. ☐

The factor $(q-\zeta)$ occurs with multiplicity $s := (p^i - i) + \sum_{d=1}^{i} p^d$ in the denominator of $E_\alpha^{(p^i\ell)}$. We want to show $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in (q-\zeta)^s \cdot \mathcal{U}_q(\mathfrak{g})$.

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- $\mathrm{Ad}(E_\alpha^\ell)(u)$ is independent of ζ. Varying ζ over all primitive ℓ-th roots of unity in $\mathbb{C}$, we conclude $\mathrm{Ad}(E_\alpha^\ell)(u) \in \phi_\ell \cdot \mathcal{U}_q(\mathfrak{g})$, where $\phi_\ell \in \mathbb{Z}[q]$ is the ℓ-th cyclotomic polynomial.

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It follows that $\mathrm{Ad}(E_\alpha^{p^i\ell})(u) \in \phi \cdot \mathcal{U}_q(\mathfrak{g})$, where $\phi = \prod_{j=0}^{i} (\phi_{p^j\ell})^{p^{i-j}}$.

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We have $\phi_{p^j\ell} \equiv (\phi_\ell)^{\varphi(p^j)} \mod p$ (Guerrier, 68). Then the factor $(q-\zeta)$ occurs in ϕ with multiplicity $r := p^i + \sum_{j=1}^{i} p^{i-1}(p-1)$. Check that $r \ge s$ with equality only if $i = 1$.

Practical applications:

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Theorem (D, 2008)

Assume $\ell \geq h$ *and* p *is good for* Φ. *Then* $H^{odd}(u_\zeta(\mathfrak{g}), k) = 0$ *and* $H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong k[\mathcal{N}]$. *For smaller values of* ℓ, *the* G*-module structure of* $H^\bullet(u_\zeta(\mathfrak{g}), k)$ *can be explicitly computed, provided certain cohomological vanishing results are true. (Brion and Kumar think that these vanishing results are probably true.)*

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