

Cohomology of Frobenius–Lusztig kernels of quantized enveloping algebras

Christopher Drupieski

University of Virginia

January 5, 2009

Let \mathfrak{g} be a simple complex Lie algebra with root system Φ .

Let k be an algebraically closed field of characteristic $p \neq 2$.
Assume $p \neq 3$ if Φ has type G_2 .

Let G be the corresponding simply-connected algebraic group over k .

The universal enveloping algebra $\mathbb{U}(\mathfrak{g})$ of \mathfrak{g} admits the familiar presentation by Chevalley generators and relations. It has a PBW basis

$$\left\{ F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} H_1^{b_1} \cdots H_n^{b_n} E_{\gamma_1}^{c_1} \cdots E_{\gamma_N}^{c_N} : a_i, b_i, c_i \geq 0 \right\}.$$

The universal enveloping algebra $\mathbb{U}(\mathfrak{g})$ of \mathfrak{g} admits the familiar presentation by Chevalley generators and relations. It has a PBW basis

$$\left\{ F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} H_1^{b_1} \cdots H_n^{b_n} E_{\gamma_1}^{c_1} \cdots E_{\gamma_N}^{c_N} : a_i, b_i, c_i \geq 0 \right\}.$$

The Kostant \mathbb{Z} -form $\mathbb{U}_{\mathbb{Z}}(\mathfrak{g})$ is spanned by monomials of divided powers.

$$\left\{ F_{\gamma_1}^{(a_1)} \cdots F_{\gamma_N}^{(a_N)} \binom{H_1}{b_1} \cdots \binom{H_n}{b_n} E_{\gamma_1}^{(c_1)} \cdots E_{\gamma_N}^{(c_N)} : a_i, b_i, c_i \geq 0 \right\}.$$

The universal enveloping algebra $\mathbb{U}(\mathfrak{g})$ of \mathfrak{g} admits the familiar presentation by Chevalley generators and relations. It has a PBW basis

$$\left\{ F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} H_1^{b_1} \cdots H_n^{b_n} E_{\gamma_1}^{c_1} \cdots E_{\gamma_N}^{c_N} : a_i, b_i, c_i \geq 0 \right\}.$$

The Kostant \mathbb{Z} -form $\mathbb{U}_{\mathbb{Z}}(\mathfrak{g})$ is spanned by monomials of divided powers.

$$\left\{ F_{\gamma_1}^{(a_1)} \cdots F_{\gamma_N}^{(a_N)} \binom{H_1}{b_1} \cdots \binom{H_n}{b_n} E_{\gamma_1}^{(c_1)} \cdots E_{\gamma_N}^{(c_N)} : a_i, b_i, c_i \geq 0 \right\}.$$

The hyperalgebra $\text{hy}(G)$ of G is defined by $\text{hy}(G) = \mathbb{U}_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} k$.

If we restrict $0 \leq a_i, b_i, c_i < p^r$, then we get a basis for $\text{hy}(G_r)$, the hyperalgebra of the r -th Frobenius kernel of G . The algebra $\text{hy}(G_r)$ is the Hopf-algebraic kernel of the map $F^r : \text{hy}(G) \rightarrow \text{hy}(G)$.

Definition

The quantized enveloping algebra $\mathbb{U}_q(\mathfrak{g})$ is the $k(q)$ -algebra with generators $E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1}$ ($\alpha \in \Pi$) and relations

$$K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha, \quad K_\alpha K_\beta = K_\beta K_\alpha \quad (R1)$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha, \beta)} E_\beta \quad (R2)$$

$$K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha, \beta)} F_\beta \quad (R3)$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha, \beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \quad (R4)$$

and the quantum Serre relations (R5) and (R6).

Given a choice of reduced expression for $w_0 \in W$, we can (non-uniquely) define root vectors E_γ, F_γ for all $\gamma \in \Phi^+$. Then $\mathbb{U}_q(\mathfrak{g})$ admits a PBW-type basis

$$\left\{ F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} K_{\alpha_1}^{b_1} \cdots K_{\alpha_n}^{b_n} E_{\gamma_1}^{c_1} \cdots E_{\gamma_N}^{c_N} : a_i, c_i \geq 0, b_i \in \mathbb{Z} \right\}.$$

Given a choice of reduced expression for $w_0 \in W$, we can (non-uniquely) define root vectors E_γ, F_γ for all $\gamma \in \Phi^+$. Then $\mathbb{U}_q(\mathfrak{g})$ admits a PBW-type basis

$$\left\{ F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} K_{\alpha_1}^{b_1} \cdots K_{\alpha_n}^{b_n} E_{\gamma_1}^{c_1} \cdots E_{\gamma_N}^{c_N} : a_i, c_i \geq 0, b_i \in \mathbb{Z} \right\}.$$

Let $\mathcal{Z} = k[q, q^{-1}]$. There exists a \mathcal{Z} -form $U_q(\mathfrak{g})$ in $\mathbb{U}_q(\mathfrak{g})$ generated by divided powers. Given $\zeta \in k^\times$, set $U_\zeta(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes_{\mathcal{Z}} k$.

We will typically assume $\zeta \in k$ to be a primitive ℓ -th root of unity.

There exists a surjective Hopf algebra homomorphism $F_\zeta : U_\zeta(\mathfrak{g}) \rightarrow \text{hy}(G)$. The Hopf-algebraic kernel $u_\zeta(\mathfrak{g})$ of F_ζ is a finite-dimensional Hopf subalgebra of $U_\zeta(\mathfrak{g})$, the (first) Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$. So

$$\text{hy}(G) \cong U_\zeta(\mathfrak{g}) // u_\zeta(\mathfrak{g}).$$

The subalgebra $u_\zeta(\mathfrak{g})$ is generated by $\{E_\alpha, F_\alpha, K_\alpha : \alpha \in \Pi\}$.

There exists a surjective Hopf algebra homomorphism $F_\zeta : U_\zeta(\mathfrak{g}) \rightarrow \text{hy}(G)$. The Hopf-algebraic kernel $u_\zeta(\mathfrak{g})$ of F_ζ is a finite-dimensional Hopf subalgebra of $U_\zeta(\mathfrak{g})$, the (first) Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$. So

$$\text{hy}(G) \cong U_\zeta(\mathfrak{g}) // u_\zeta(\mathfrak{g}).$$

The subalgebra $u_\zeta(\mathfrak{g})$ is generated by $\{E_\alpha, F_\alpha, K_\alpha : \alpha \in \Pi\}$.

If $p = \text{char}(k) > 0$, then we have $F^r \circ F_\zeta : U_\zeta(\mathfrak{g}) \rightarrow \text{hy}(G)$. The Hopf-algebraic kernel of $F^r \circ F_\zeta$ is a finite-dimensional subalgebra $U_\zeta(G_r)$ of $U_\zeta(\mathfrak{g})$, the r -th Frobenius–Lusztig kernel of $U_\zeta(\mathfrak{g})$. The algebra $U_\zeta(G_r)$ is generated by

$$\left\{ E_\alpha, E_\alpha^{(p^i \ell)}, F_\alpha, F_\alpha^{(p^i \ell)}, K_\alpha : \alpha \in \Pi, 0 \leq i \leq r-1 \right\}.$$

We have $\text{hy}(G_r) \cong U_\zeta(G_r) // u_\zeta(\mathfrak{g})$.

- 1 What does $H^\bullet(u_\zeta(\mathfrak{g}), k)$ look like (in all characteristics)?

- ① What does $H^\bullet(u_\zeta(\mathfrak{g}), k)$ look like (in all characteristics)?
- $H^\bullet(G_1, k) \cong k[\mathcal{N}]$ if $p \geq h$, by independent work of Friedlander-Parshall (1986) and Andersen-Jantzen (1984).
 - $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$ if $\ell \geq h$ by Ginzburg-Kumar (1993); computed for $\ell < h$ by Bendel-Nakano-Parshall-Pillen (2007).

- ① What does $H^\bullet(u_\zeta(\mathfrak{g}), k)$ look like (in all characteristics)?
 - $H^\bullet(G_1, k) \cong k[\mathcal{N}]$ if $p \geq h$, by independent work of Friedlander-Parshall (1986) and Andersen-Jantzen (1984).
 - $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$ if $\ell \geq h$ by Ginzburg-Kumar (1993); computed for $\ell < h$ by Bendel-Nakano-Parshall-Pillen (2007).
- ② What does $H^\bullet(U_\zeta(G_r), k)$ look like? ($p > 0$)

- ① What does $H^\bullet(u_\zeta(\mathfrak{g}), k)$ look like (in all characteristics)?
 - $H^\bullet(G_1, k) \cong k[\mathcal{N}]$ if $p \geq h$, by independent work of Friedlander-Parshall (1986) and Andersen-Jantzen (1984).
 - $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$ if $\ell \geq h$ by Ginzburg-Kumar (1993); computed for $\ell < h$ by Bendel-Nakano-Parshall-Pillen (2007).
- ② What does $H^\bullet(U_\zeta(G_r), k)$ look like? ($p > 0$)
 - $H^\bullet(G_1, k)$ is finitely-generated (for all p); $H^\bullet(U_r, k)$ and $H^\bullet(B_r, k)$ are finitely-generated (for all r) by Friedlander-Parshall (1986).
 - $H^\bullet(H, k)$ is finitely-generated for any finite group scheme H by Friedlander-Suslin (1997).

Theorem

Assume ℓ is odd and coprime to 3 if Φ has type G_2 . Assume $p = \text{char}(k)$ is good for Φ . Assume $\ell \neq 9$ if Φ has type E_6 , and $\ell \nmid n + 1$ if Φ has type A_n . Then there exists $J \subset \Pi$ (depending on ℓ) such that

$$H^{\text{odd}}(u_\zeta(\mathfrak{g}), k) = 0, \quad H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \text{ind}_{P_J}^G S^\bullet(u_J^*),$$

as G -modules, provided $R^i \text{ind}_{P_J}^G S^\bullet(u_J^*) = 0$ for all $i > 0$.

Theorem

Assume ℓ is odd and coprime to 3 if Φ has type G_2 . Assume $p = \text{char}(k)$ is good for Φ . Assume $\ell \neq 9$ if Φ has type E_6 , and $\ell \nmid n + 1$ if Φ has type A_n . Then there exists $J \subset \Pi$ (depending on ℓ) such that

$$H^{\text{odd}}(u_\zeta(\mathfrak{g}), k) = 0, \quad H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \text{ind}_{P_J}^G S^\bullet(u_J^*),$$

as G -modules, provided $R^i \text{ind}_{P_J}^G S^\bullet(u_J^*) = 0$ for all $i > 0$.

Proof.

Similar to the classical argument for $H^\bullet(G_1, k)$. Same strategy as used by BNPP for the case $k = \mathbb{C}$, but some additional calculations required when $p = \text{char}(k) > 0$. (The vanishing condition is known to hold for some $J \subset \Pi$, and is suspected to hold in general.) □

Theorem

Assume ℓ is odd, coprime to 3 if Φ has type G_2 , and $\ell \geq h$. Assume p is good for Φ . Then there exists an isomorphism of G -algebras

$$H^{2\bullet}(u_\zeta(\mathfrak{g}), k) \cong \text{ind}_B^G S^\bullet(\mathfrak{u}^*) \cong k[\mathcal{N}].$$

Theorem

Assume that one of the following conditions is satisfied:

- 1 Φ is of type A_n or type D_n , and $n > p + 2$,
- 2 Φ is of type A_n and $\ell \geq n + 4$, or
- 3 Φ is of type D_n and $\ell \geq 4n$.

Then $H^\bullet(U_\zeta(G_1), k)$ is a Noetherian algebra.

Theorem

Assume that one of the following conditions is satisfied:

- ① Φ is of type A_n or type D_n , and $n > p + 2$,
- ② Φ is of type A_n and $\ell \geq n + 4$, or
- ③ Φ is of type D_n and $\ell \geq 4n$.

Then $H^\bullet(U_\zeta(G_1), k)$ is a Noetherian algebra.

Proof.

There exists a spectral sequence converging to $H^\bullet(U_\zeta(G_1), k)$ with

$$E_2^{i,j} = H^i(U_\zeta(G_1) // u_\zeta(\mathfrak{g}), H^j(u_\zeta(\mathfrak{g}), k)) \cong H^i(G_1, H^j(G_1, k)^{(-1)})$$

Now $R = E_2^{0,2p}$ consists of permanent cycles, hence $E_2^{\bullet,0} = H^\bullet(G_1, k)$ and R generate a Noetherian subalgebra of permanent cycles over which $E_2^{\bullet,\bullet}$ is a Noetherian module. □

Theorem

Assume ℓ is odd and coprime to 3 if Φ has type G_2 . Let $r \geq 1$. Then $H^\bullet(U_\zeta(U_r), k)$ is a Noetherian algebra.

Theorem

Assume ℓ is odd and coprime to 3 if Φ has type G_2 . Let $r \geq 1$. Then $H^\bullet(U_\zeta(U_r), k)$ is a Noetherian algebra.

Proof.

There exists a multiplicative filtration on $U_\zeta(U_r)$ such that $H^\bullet(\text{gr } U_\zeta(U_r), k) \cong \Lambda_{\zeta, r}^\bullet \otimes S^\bullet(R^*)$. In the spectral sequence

$$E_2^{i,j} = H^{i+j}(\text{gr } U_\zeta(U_r), k)_{(i)} \Rightarrow H^{i+j}(U_\zeta(U_r), k),$$

the subalgebra $S^\bullet(R^*)$ consists of permanent cycles. □

Theorem

Assume ℓ is odd and coprime to 3 if Φ has type G_2 . Let $r \geq 1$. Then $H^\bullet(U_\zeta(U_r), k)$ is a Noetherian algebra.

Proof.

There exists a multiplicative filtration on $U_\zeta(U_r)$ such that $H^\bullet(\text{gr } U_\zeta(U_r), k) \cong \Lambda_{\zeta,r}^\bullet \otimes S^\bullet(R^*)$. In the spectral sequence

$$E_2^{i,j} = H^{i+j}(\text{gr } U_\zeta(U_r), k)_{(i)} \Rightarrow H^{i+j}(U_\zeta(U_r), k),$$

the subalgebra $S^\bullet(R^*)$ consists of permanent cycles. □

Corollary

$H^\bullet(U_\zeta(B_r), k) = H^\bullet(U_\zeta(U_r), k)^{U_\zeta(T_r)}$ is Noetherian.

① Is $H^\bullet(U_\zeta(G_1), k)$ always a Noetherian algebra?

- 1 Is $H^\bullet(U_\zeta(G_1), k)$ always a Noetherian algebra?
- 2 Is $H^\bullet(U_\zeta(G_r), k)$ always a Noetherian algebra?