Homological Algebra

Based on *Basic Homological Algebra*
by M. Scott Osborne

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2 Modules

Proposition. Suppose $A_1, A_2, A \in_R M$, and suppose $A_1 \overset{\pi_1}{\leftarrow} A \overset{\pi_2}{\rightarrow} A_2$ and $A_1 \overset{\varphi_1}{\rightarrow} A \overset{\varphi_2}{\leftarrow} A_2$ are morphisms satisfying $\pi_1\varphi_1 = i_{A_1}$, $\pi_2\varphi_2 = i_{A_2}$, and $\varphi_1\pi_1 + \varphi_2\pi_2 = i_A$. Then $\varphi_2\pi_1 = 0$, $\varphi_1\pi_2 = 0$, and A is both a product and a coproduct of A_1 and A_2.

We have the following isomorphisms:

$$\begin{array}{rcl} \mathrm{Hom}(A, \prod B_i) & \cong & \prod \mathrm{Hom}(A, B_i) \\ \mathrm{Hom}(\oplus A_i, B) & \cong & \prod \mathrm{Hom}(A_i, B) \end{array}$$

Proposition. Suppose $B \in_R M$. Then

1. $R \otimes B \cong B$ as abelian groups.

2. $(R/I) \otimes B \cong B/IB$ if $I \subseteq R$ is a right ideal.

Theorem. Suppose $A \in M_R$, $B \in {}_RM$, and $G \in \mathbf{Ab}$. Then, as abelian groups,

$$\mathrm{Hom}_{\mathbb{Z}}(A \otimes B, G) \cong \mathrm{Hom}_R\left(B, \mathrm{Hom}_{\mathbb{Z}}(A, G)\right)$$

This isomorphism is natural in A, B adn G.

Proposition (5-Lemma)**.** Suppose

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_4} & & \downarrow{\scriptstyle\varphi_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

is commutative with exact rows, and suppose

1. φ_2 and φ_4 are isomorphisms

2. φ_1 is onto, and

3. φ_5 is injective.

Then φ_3 is an isomorphism.

Proposition.

1. If $A \in_R M$, then $\mathrm{Hom}(A, \bullet)$ is left exact.

2. If $A \in M_R$, then $\mathrm{Hom}(\bullet, A)$ is left exact.

3. If $A \in M_R$, then $A\otimes$ is right exact.

Definition.

1. Suppose $A \in {}_RM$. A is *projective* if $\mathrm{Hom}(A, \bullet)$ is an exact functor.

2. Suppose $A \in M_R$. A is *injective* if $\operatorname{Hom}(\bullet, A)$ is an exact functor.
3. Suppose $A \in M_R$. A is *flat* if $A\otimes$ is an exact functor.

Proposition.

1. $\bigoplus A_i$ is projective if and only if each A_i is projective.
2. $\prod A_i$ is injective if and only if each A_i is injective.
3. $\bigoplus A_i$ is flat if and only if each A_i is flat.

Proposition. Suppose $E \in {}_RM$. Then E is injective if and only if a filler g exists for every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \hookrightarrow & R \\ & & \Big\downarrow f & \swarrow g & \\ & & E & & \end{array}$$

where I is a left ideal in R.

Recall that E is divisible if for every right nonzero divisor a, $aE = E$.

Corollary. Suppose R is a PID, and suppose $E \in {}_RM$ is divisible. Then E is injective.

Theorem. Suppose $A \in M_R$ is flat, and suppose $G \in {}_{\mathbb{Z}}M$ is injective. Then $\operatorname{Hom}_{\mathbb{Z}}(A, G)$ is injective in ${}_RM$.

Corollary (Enough Injectives)**.** If $A \in {}_RM$, then there exists an injective $E \in {}_RM$ and an injection $A \hookrightarrow E$.

E is injective if and only if E is an absolute direct summand.

Proposition. R is left Noetherian if and only if every direct sum of injectives in ${}_RM$ is injective.

Exercise. Suppose F is an exact functor. Then F takes any exact sequence $A \to B \to C$ to an exact sequence.

Exercise. Any injective module is divisible.

Exercise (Flat test lemma)**.** Suppose $A \in M_R$. Then A is flat if and only if $A \otimes I \to AI$ is one-to-one for every finitely generated left ideal I.

Exercise. Suppose R is a PID. Then A is flat if and only if A is torsion free.

Exercise (Short 5-Lemma)**.** Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \Big\downarrow \eta & & \Big\downarrow \psi & & \Big\downarrow \phi & & \\ 0 & \longrightarrow & A' & \xrightarrow{j'} & B' & \xrightarrow{\eta'} & C' & \longrightarrow & 0 \end{array}$$

in ${}_RM$ with exact rows. Then

1. If η and ϕ are one-to-one, then so is ψ.
2. If η and ϕ are onto, then so is ψ.

3 Ext and Tor

Call the sequence

$$A \xrightarrow{d} B \xrightarrow{\partial} C$$

a *complex* if $\partial d = 0$. The *homology* of the complex is $\ker \partial / \operatorname{im} d$. Suppose

$$\begin{array}{ccccc} A & \xrightarrow{d} & B & \xrightarrow{\partial} & C \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \eta} \\ A' & \xrightarrow{d'} & B' & \xrightarrow{\partial'} & C' \end{array}$$

commutes, with rows that are complexes. Set $H = \ker \partial / \operatorname{im} d$, and $H' = \ker \partial' / \operatorname{im} d'$. Define $\psi_* : H \to H'$ by $\psi_*(x + \operatorname{im} d) = \psi(x) + \operatorname{im} d'$. Suppose also that

$$\begin{array}{ccccc} A & \xrightarrow{d} & B & \xrightarrow{\partial} & C \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \eta'} \\ A' & \xrightarrow{d'} & B' & \xrightarrow{\partial'} & C' \end{array}$$

also commutes. then $\psi_* = \psi'_*$ if ψ, ψ' are *chain homotopic*, that is, if there exist maps $D : B \to A'$ and $\Delta : C \to B'$ such that $\psi - \psi' = d'D + \Delta\partial$.

Proposition. Suppose $B, B' \in {}_RM$, and $\varphi \in \operatorname{Hom}(B, B')$. Suppose $\langle P_n, d_n \rangle$ is a projective resolution of B, and $\langle P'_n, d'_n \rangle$ is a projective resolution of B'. Then there exist fillers $\varphi_n \in \operatorname{Hom}(P_n, P'_n)$ making

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} \cdots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \\ \cdots & \longrightarrow & P'_{n+1} & \xrightarrow{d'_{n+1}} & P'_n & \xrightarrow{d'_n} \cdots \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\pi'} & B' & \longrightarrow & 0 \end{array}$$

commutative. Further, if $\varphi'_n \in \operatorname{Hom}(P_n, P'_n)$ are also fillers, than φ_n and φ'_n are homotopic.

Definition. Let $A \in M_R$, $B, C \in {}_RM$.

1. Apply $A\otimes$ to any projective resolution of B. The n-th homology of this complex, $\ker(A \otimes d_n) / \operatorname{im}(A \otimes d_{n+1})$, is (isomorphic to) $\operatorname{Tor}_n^R(A, B)$.

2. Apply $\operatorname{Hom}_R(\bullet, C)$ to any projective resolution of B. The n-th homology of this complex is $\operatorname{Ext}_R^n(B, C)$.

Proposition. If $A \in M_R$, $B, C \in {}_RM$, then

1. $\operatorname{Tor}_0(A, B) \cong A \otimes B$

2. $\operatorname{Ext}^0(B, C) \cong \operatorname{Hom}(B, C)$

3. $\operatorname{Tor}_n(A, B) = 0$ if A is flat or B is projective, $n \geq 1$.

4. $\text{Ext}^n(B, C) = 0$ if B is projective of C is injective, $n \geq 1$.

Theorem. Suppose

$$0 \to \mathcal{C} \xrightarrow{\varphi} \mathcal{C}' \xrightarrow{\psi} \mathcal{C}'' \to 0$$

is a short exact sequence of chain complexes. Then there exist maps $\delta_n : H_n(\mathcal{C}'') \to H_{n-1}(\mathcal{C})$ such that

$$\cdots \to H_{n+1}(\mathcal{C}'') \xrightarrow{\delta_{n+1}} H_n(\mathcal{C}) \xrightarrow{H_n(\varphi)} H_n(\mathcal{C}') \xrightarrow{H_n(\psi)} H_n(\mathcal{C}'') \xrightarrow{\delta_n} \cdots$$

is exact. Moreoever, the sequence of maps is natural.

Theorem.

1. (First Long Exact Sequence for Tor) Suppose $0 \to A \to A' \to A'' \to 0$ is short exact in M_R. Then for all $B \in {}_RM$, there is a long exact sequence

$$\cdots \to \text{Tor}_{n+1}(A'', B) \xrightarrow{\delta_{n+1}} \text{Tor}_n(A, B) \to \text{Tor}_n(A', B) \to \text{Tor}_n(A'', B) \xrightarrow{\delta_n} \text{Tor}_{n-1}(A, B) \to$$

$$\cdots \to A \otimes B \to A' \otimes B \to A'' \otimes B \to 0$$

2. (First Long Exact Sequence for Ext) Suppose $0 \to C \to C' \to C'' \to 0$ is short exact in ${}_RM$. Then for all $B \in {}_RM$, there is a long exact sequence

$$\cdots \leftarrow \text{Ext}^{n+1}(B, C) \leftarrow \text{Ext}^n(B, C'') \leftarrow \text{Ext}^n(B, C') \leftarrow \text{Ext}^n(B, C) \leftarrow \text{Ext}^{n-1}(B, C'') \leftarrow$$

$$\cdots \leftarrow \text{Hom}(B, C'') \leftarrow \text{Hom}(B, C') \leftarrow \text{Hom}(B, C) \leftarrow 0$$

Corollary. Suppose $0 \to A \to F \to A' \to 0$ is short exact in M_R with F flat. Then $\text{Tor}_n(A, B) \cong \text{Tor}_{n+1}(A', B)$ for all $b \in {}_RM$ and $n \geq 1$.

Corollary. Suppose $0 \to C \to E \to C' \to 0$ is short exact in ${}_RM$ with E injective. Then $\text{Ext}^n(B, C') \cong \text{Ext}^{n+1}(B, C)$ for all $B \in {}_RM$ and $n \geq 1$.

Corollary. Suppose $B \in {}_RM$ and suppose $\text{Tor}_1(R/I, B) = 0$ for every finitely generated right ideal I. Then B is flat.

Corollary. Suppose $B \in {}_RM$. The following are equivalent:

1. B is projective.

2. For all $C \in {}_RM$ and $n \geq 1$, $\text{Ext}^n(B, C) = 0$.

3. For all $c \in {}_RM$, $\text{Ext}^1(B, C) = 0$.

Theorem. $\text{Tor}_n(A, B)$ is isomorphic of the n-th homology of a flat resolution of A, tensored with B (and $A \otimes B$ deleted). Futhermore, the isomorphism is natural in that if $\varphi \in \text{Hom}(B, B')$, the an appropriate diagram involving $\text{Tor}_n(A, \varphi)$ commutes.

Theorem. $\text{Ext}^n(B, C)$ is isomorphic to the n-th homology of $\text{Hom}(B, \bullet)$ applied to an injective resolution of C (and $\text{Hom}(B, C)$ deleted). Furthermore, the isomorphism is natural in that if $\varphi \in \text{Hom}(B, B')$, then an appropriate diagram involving $\text{Ext}^n(\varphi, C)$ commutes.

Proposition (Second Long Exact Sequence for Ext)**.** Suppose $0 \to B \to B' \to B'' \to 0$ is short exact in ${}_RM$, and suppose $C \in {}_RM$. Then there is a long exact sequence

$$\cdots \leftarrow \text{Ext}^{n+1}(B'', C) \leftarrow \text{Ext}^n(B, C) \leftarrow \text{Ext}^n(B', C) \leftarrow \text{Ext}^n(B'', C) \leftarrow \text{Ext}^{n-1}(B, C) \leftarrow$$

$$\cdots \leftarrow \text{Hom}(B, C) \leftarrow \text{Hom}(B', C) \leftarrow \text{Hom}(B'', C) \leftarrow 0$$

Corollary. Suppose $0 \to B \to P \to B'' \to 0$ is short exact in ${}_RM$, with P projective. Then $\text{Ext}^n(B, C) \cong \text{Ext}^{n+1}(B', C)$ for all $C \in {}_RM$ and $n \geq 1$.

Corollary. Suppose $C \in {}_RM$. The following are equivalent:

1. C is injective.
2. $\text{Ext}^n(B, C) = 0$ for all $B \in {}_RM$ and $n \geq 1$.
3. $\text{Ext}^1(R/I, C) = 0$ for all left ideals I.

Example. $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}$ (as groups).

Proposition. $\text{Tor}_n^R(A, B) \cong \text{Tor}_n^{R^{op}}(B, A)$

Corollary (Second Long Exact Sequence for Tor)**.** If $0 \to B \to B' \to B'' \to 0$ is short exact, then there is a long exact sequence

$$\cdots \to \text{Tor}_{n+1}(A, B'') \xrightarrow{\delta_{n+1}} \text{Tor}_n(A, B) \to \text{Tor}_n(A, B') \to \text{Tor}_n(A, B'') \xrightarrow{\delta_n} \text{Tor}_{n-1}(A, B) \to$$

$$\cdots \to A \otimes B \to A \otimes B' \to A'' \otimes B'' \to 0$$

Corollary. Suppose $A \in M_R$. Then the following are equivalent:

1. A is flat.
2. $\text{Tor}_n^R(A, B) = 0$ for all $B \in {}_RM$ and $n \geq 1$.
3. $\text{Tor}_1^R(A, R/I) = 0$ for every finitely generated left ideal I.

Corollary. Suppose $B \in {}_RM$. The following are equivalent:

1. B is flat.
2. $\text{Tor}_n^R(A, B) = 0$ for all $A \in M_R$ and $n \geq 1$.
3. $\text{Tor}_1^R(R/J, B) = 0$ for every finitely generated right ideal J.

Corollary. Suppose $0 \to B \to F \to B' \to 0$ is short exact in ${}_RM$ with F flat. Then $\text{Tor}_n(A, B) \cong \text{Tor}_{n+1}(A, B')$ for all $A \in M_R$ and $n \geq 1$.

Corollary. $\text{Tor}_n^R(A, B)$ can be computed from any flat resolution of B.

Exercise. If $\text{Ext}^1_R(B, C) = 0$, then any short exact sequence $0 \to C \to X \to B \to 0$ splits.

Exercise. Suppose I is a left ideal and J is a right ideal. Then

1. $\operatorname{Tor}_n(R/J, R/I) \cong \operatorname{Tor}_{n-2}(J, I)$ for $n > 2$.
2. $\operatorname{Tor}_2(R/J, R/I) \cong \ker(J \otimes I \to JI)$
3. $\operatorname{Tor}_1(R/J, R/I) \cong (J \cap I)/(JI)$.

Exercise. Suppose B is an abelian group, and let $T(B)$ denote its torsion subgroup (subgroup of elements of finite order). Then $T(B) \cong \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$.

Exercise.

1. $\operatorname{Ext}^n(\oplus_I B_i, C) \cong \prod_I \operatorname{Ext}^n(B_i, C)$
2. $\operatorname{Ext}^n(B, \prod_I C_i) \cong \prod_I \operatorname{Ext}^n(B, C_i)$
3. $\operatorname{Tor}_n(A, \oplus_I B_i) \cong \bigoplus_I \operatorname{Tor}_n(A, B_i)$

Exercise (Algebraic Mayer-Vietoris Sequence)**.** Suppose B_1, B_2 are submodules of $B \in {}_RM$. Let $C \in {}_RM$. Then there exists a long exact sequence

$$\leftarrow \operatorname{Ext}^{n+1}(B_1+B_2, C) \leftarrow \operatorname{Ext}^n(B_1 \cap B_2, C) \leftarrow \operatorname{Ext}^n(B_1, C) \oplus \operatorname{Ext}^n(B_2, C) \leftarrow \operatorname{Ext}^n(B_1+B_2, C) \leftarrow$$

$$\cdots \leftarrow \operatorname{Hom}(B_1 \cap B_2, C) \leftarrow \operatorname{Hom}(B_1, C) \oplus \operatorname{Hom}(B_2, C) \leftarrow \operatorname{Hom}(B_1 + B_2, C) \leftarrow 0$$

4 Dimension Theory

Definition.

1. Let $B \in {}_RM$. The *projective dimension* of B is

$$P\text{–}\dim B = \inf \left\{ n \geq 0 : \operatorname{Ext}^{n+1}(B, \bullet) \equiv 0 \right\}$$

2. Let $C \in {}_RM$. The *injective dimension* of C is

$$I\text{–}\dim C = \inf \left\{ n \geq 0 : \operatorname{Ext}^{n+1}(\bullet, C) \equiv 0 \right\}$$

3. Let $B \in {}_RM$. The *flat dimension* of B is

$$F\text{–}\dim B = \inf \{ n \geq 0 : \operatorname{Tor}_{n+1}(\bullet, B) \equiv 0 \}$$

4. The *left global dimension* of R is

$$LG\text{–}\dim R = \sup \{ P\text{–}\dim B : B \in {}_RM \}$$

5. The *right global dimension* of R is

$$RG\text{–}\dim R = \sup \{ P\text{–}\dim A : A \in M_R \}$$

6. The *(left) weak dimension* of R is

$$W\text{–}\dim R = \sup F\text{–}\dim B : B \in {}_RM$$

Proposition.

1. $LG\text{–}\dim R = \inf \left\{ n \geq 0 : \operatorname{Ext}^{n+1}(\bullet, \bullet) \equiv 0 \right\} = \sup \{ I\text{–}\dim C : C \in {}_RM \}$.
2. $W\text{–}\dim R = \inf \{ n \geq 0 : \operatorname{Tor}_{n+1}(\bullet, \bullet) \equiv 0 \} = \sup \{ F\text{–}\dim A : A \in M_R \}$.

Proposition. Suppose $0 \to D \to L_1 \to L_2 \to \cdots \to L_n \to D' \to 0$ is exact in ${}_RM$ and $d \geq 0$.

1. If $P\text{–}\dim L_j \leq d$ for all j, then $\operatorname{Ext}^k(D, C) \cong \operatorname{Ext}^{k+n}(D', C)$ for all $C \in {}_RM$ and $k > d$.
2. If $I\text{–}\dim L_j \leq d$ for all j, then $\operatorname{Ext}^k(B, D') \cong \operatorname{Ext}^{k+1}(B, D)$ for all $B \in {}_RM$ and $k > d$.
3. If $F\text{–}\dim L_j \leq d$ for all j, then $\operatorname{Tor}_k(A, D) \cong \operatorname{Tor}_{k+n}(A, D')$ for all $A \in M_R$ and $k > d$.

Corollary. Suppose $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to B \to 0$ is short exact in ${}_RM$.

1. If $P\text{–}\dim Q_j \leq d$ for all j, then $P\text{–}\dim B \leq d + n$.
2. If $F\text{–}\dim Q_j \leq d$ for all j, then $F\text{–}\dim B \leq d + n$.

Given a projective (or flat) resolution $\langle P_k, d_k\rangle$ of $B \in {}_RM$ and projection $\pi : P_0 \to B$, set $K_0 = B$, $K_1 = \ker \pi$ and $K_n = \ker d_{n-1}$ if $n \geq 2$. Call K_n the *n-th kernel* of the projective (or flat) resolution $\langle P_k, d_k\rangle$.

Proposition (Projective Dimension Theorem)**.** Suppose $B \in {}_RM$. The following are equivalent:

1. $P\text{−}\dim B \leq n$.
2. The n-th kernel of any projective resolution of B is projective.
3. There exists a projective resolution of B whose n-th kernel is projective.
4. There exists a projective resolution $\langle P_k, d_k\rangle$ of B for which $P_k = 0$ when $k > n$.

Proposition (Flat Dimension Theorem)**.** Suppose $B \in {}_RM$. The following are equivalent:

1. $F\text{−}\dim B \leq n$.
2. $\mathrm{Tor}_{n+1}(R/I, B) = 0$ for all finitely generated right ideals I.
3. The n-th kernel of any flat resolution of B is flat.
4. There exists a flat resolution of B whose n-th kernel is flat.
5. There exists a flat resolution $\langle F_k, d_k\rangle$ of B for which $F_k = 0$ when $k > n$.

Corollary. For all $B \in {}_RM$, $F\text{−}\dim B \leq P\text{−}\dim B$.

Corollary. $LG\text{−}\dim R \geq W\text{−}\dim R$ and $RG\text{−}\dim R \geq W\text{−}\dim R$.

Given an injective resolution $\langle E_k, d_k\rangle$ of $C \in {}_RM$ and inclusion $\iota : C \to E_0$, set $D_0 = C$ and $D_n = \operatorname{im} d_n$ if $n \geq 1$. Call D_n the *n-th cokernel* of the injective resolution.

Proposition (Injective Dimension Theorem)**.** Suppose $C \in {}_RM$. The following are equivalent:

1. $I\text{−}\dim C \leq n$.
2. $\mathrm{Ext}^{n+1}(R/I, C) = 0$ for all left ideals I.
3. The n-th cokernel of any injective resolution of C is injective.
4. There exists an injective resolution of C whose n-th cokernel is injective.
5. There exists an injective resolution $\langle E_k, d_k\rangle$ of C for which $E_k = 0$ when $k > n$.

Proposition (Global Dimension Theorem)**.**

$$LG\text{−}\dim R = \sup\{P\text{−}\dim(R/I) : I \text{ a left ideal}\}$$

Corollary. If $LG\text{−}\dim R > 0$, then $LG\text{−}\dim R = 1 + \sup\{P\text{−}\dim I : I \text{ a left ideal}\}$.

Corollary. $LG\text{-}\dim R \leq 1$ if and only if every left ideal is projective.

Corollary. If R is a PID, then $LG\text{-}\dim R \leq 1$.

Proposition (Weak Dimension Theorem)**.**

$$\begin{aligned} W\text{-}\dim R &= \sup\{F\text{-}\dim(R/I) : I \text{ a finitely generated right ideal}\} \\ &= \sup\{F\text{-}\dim(R/I) : I \text{ a finitely generated left ideal}\} \end{aligned}$$

Corollary. If $W\text{-}\dim R > 0$, then

$$\begin{aligned} W\text{-}\dim R &= 1 + \sup\{F\text{-}\dim I : I \text{ a finitely generated right ideal}\} \\ &= 1 + \sup\{F\text{-}\dim I : I \text{ a finitely generated left ideal}\} \end{aligned}$$

Suppose $B \in {}_RM$, and set $B^* = \operatorname{Hom}_R(B, R)$. Then $B^* \in M_R$. Let $C \in {}_RM$. Then we have a natural map $B^* \otimes_R C \to \operatorname{Hom}_R(B, C)$ that satisfies $f \otimes c \mapsto \varphi$, where $\varphi(b) = f(b)c$.

Proposition (Projective Basis Theorem)**.** Suppose $P \in {}_RM$. The following are equivalent:

1. P is projective.
2. If P is generated by $\{s_i : i \in I\}$, then there exist $\varphi_i \in P^*, i \in I$, such that for all $x \in P$, $\{i \in I : \varphi_i(x) \neq 0\}$ is finite, and $x = \sum \varphi_i(x)s_i$.
3. There exists a generating set $\{s_i : i \in I\}$ of P for which there exist $\varphi_i \in P^*, i \in I$, such that for all $x \in P$, $\{i \in I : \varphi_i(x) \neq 0\}$ is finite, and $x = \sum \varphi_i(x)s_i$.

Corollary. Suppose P is finitely generated. Then P is projective if and only if the image of the natural map $P^* \otimes P \to \operatorname{Hom}(P, P)$ contains i_p.

Note that $R^* \cong R$, so that $R^* \otimes C \cong C \cong \operatorname{Hom}_R(R, C)$ for any $C \in {}_RM$. then for any finitely generated free R-module F, $F^* \otimes C \cong \operatorname{Hom}(F, C)$.

Suppose $B \in {}_RM$ is finitely generated. B is finitely presented provided there exists a free module F and a surjective map $\pi : F \to B$ such that $\ker \pi$ is also finitely generated.

Proposition. Suppose $B \in {}_RM$ is flat, and suppose $C \in {}_RM$ is finitely presented. Then $C^* \otimes B \to \operatorname{Hom}(C, B)$ is an isomorphism.

Theorem. Suppose $P \in {}_RM$ is finitely generated. The following are equivalent:

1. P is projective.
2. P is flat and finitely presnted.
3. The natural map from $P^* \otimes P$ to $\operatorname{Hom}(P, P)$ is an isomorphism.
4. The image of the natural map from $P^* \otimes P$ to $\operatorname{Hom}(P, P)$ contains i_P.

If R is left Noetherian and $B \in {}_RM$ is finitely generated, then B has a projective resolution consisting of finitely generated free modules. The n-th kernel of such a resolution will be finitely presented. Hence, it will be projective exactly when it is flat.

Proposition. Suppose R is left Noetherian, and suppose B is a finitely generated left R-module. Then $P\text{–}\dim B = F\text{–}\dim B$.

Corollary. Suppose R is left Noetherian. Then $LG\text{–}\dim R = W\text{–}\dim R$.

Corollary. Suppose R is both right and left Noetherian. Then $LG\text{–}\dim R = RG\text{–}\dim R$.

Call an integral domain R a Dedekind domain if R has global dimension ≤ 1. Every ideal I in a Dedekind domain is projective.

Proposition. Suppose R is commutative, and I is a projective ideal containing a nonzero divisor b. Then I is finitely generated, say by $s_1, \ldots, s_n$. Further, there exist $b_1, \ldots, b_n \in R$ such that, for all j, $B \mid xb_j$ for all $x \in I$, and $x = \sum ((xb_j)/b)\, s_j$. In particular, if R is an integral domain, than any projective ideal is finitely generated; hence, any Dedekind domain is Noetherian.

Theorem (Artin–Wedderburn Structure Theorem)**.** Suppose R is a ring. The following are equivalent:

1. $LG\text{–}\dim R = 0$.
2. Every left R-module is projective.
3. Every right R-module is injective.
4. Every left R-module is semisimple.
5. Every short exact sequence of left R-modules splits.
6. Every left ideal is injective.
7. Every maximal left ideal is injective.
8. Every maximal left ideal is a direct summand of R.
9. For every left ideal I, R/I is projective.
10. Every simple left R-module is projective.
11. R is semisimple as a left R-module.
12. R is a finite direct sum of matrix rings over division rings.

Definition (von Neumann)**.** If R is a ring, then R is *regular* if, for all $a \in R$, there exists $r \in R$ such that $a = ara$.

Lemma. Suppose R is a ring an I is a left ideal. Then I is a direct summand of R if and only if I is principal and generated by an idempotent.

Lemma. Suppose R is regular. Then every finitely generated left ideal is principal (and generated by an idempotent).

A regular integral domain is a field.

Theorem (Weak Dimension Zero Characterization)**.** Suppose R is a ring. The following conditions are equivalent:

1. $W\text{--}\dim R = 0$
2. Every left R module is flat.
3. For every finitely generated left ideal I, R/I is projective.
4. $\mathrm{Tor}_1(R/J, R/I) = 0$ for every finitely generated right ideal J and every finitely generated left ideal I.
5. $\mathrm{Tor}_1(R/aR, R/Ra)$ for every $a \in R$.
6. R is regular.

An integral domain in which every finitely generated ideal is principal is called a *Bezout domain*. Suppose R is a Bezout domain. Then every principal ideal in R is isomorphic to R, hence projective, hence flat. Thus a Bezout domain has weak dimension less than or equal to one. Any Bezout domain which is not a field has weak dimension one.

Exercise. Suppose $0 \to B \to B' \to B'' \to 0$ is short exact in ${}_RM$ and suppose $P\text{--}\dim B > P\text{--}\dim B'$ or $P\text{--}\dim B'' > 1 + P\text{--}\dim B'$. Then $P\text{--}\dim B'' = 1 + P\text{--}\dim B$.

Exercise. If $0 \to K_i \to P_i \to B \to 0$ are short exact for $i = 1, 2$, with P_i projective, then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Exercise. Suppose B is finitely presented, and suppose P is projective and finitely generated, with $0 \to K \to P \to B \to 0$ short exact. Then K is finitely generated.

Exercise. A ring R is Boolean if $x = x^2$ for all $x \in R$.

1. Any Boolean ring R is commutative.
2. Any Boolean ring is regular.
3. Any finite Boolean ring is isomorphic to a (finite) direct sum of copies of $\mathbb{Z}_2$.

Exercise. Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$. Then R is a Boolean ring, hence regular. Let $I = \oplus_{i=1}^{\infty} \mathbb{Z}_2$.

1. I is projective but not finitely generated.
2. R/I is flat and finitely generated, but neither finitely presented nor projective.
3. $LG\text{--}\dim R > W\text{--}\dim R$.

Exercise. Suppose P is projective and finitely generated in ${}_RM$, and suppose $C \in {}_RM$. Then $P^* \otimes C \to \mathrm{Hom}(P, C)$ is an isomorphism.

Exercise. Suppose $P\text{--}\dim B = N \geq n$. Then the n-th kernel of any projective resolution of B has projective dimension $N - n$.

5 Change of Rings

Theorem. Suppose $F : {}_SM \to {}_RM$ is an exact, strongly additive covariant functor. Then for all $B \in {}_SM$,

1. $P\text{–}\dim_R F(B) \le P\text{–}\dim_S B + P\text{–}\dim_R F(S)$
2. $F\text{–}\dim_R F(B) \le P\text{–}\dim_S B + F\text{–}\dim_R F(S)$

If $B \in {}_RM$, and if B' is a submodule of B, define the "Supremal projective dimension" of (B', B) as follows:

$$SP\text{–}\dim(B', B) = \sup\{P\text{–}\dim C : C \supset B' \text{ is a submodule of } B\}$$

Proposition. Suppose $B \in {}_RM$, B' is a submodule of B, and B'' is a submodle of B'. Then

$$SP\text{–}\dim(B'', B) = \max\{SP\text{–}\dim(B'', B'), SP\text{–}\dim(B', B)\}$$

Corollary. If $LG\text{–}\dim R > 0$, and $0 = I_0 \supset I_1 \supset \cdots \supset I_n = R$ is a chain of left ideals in R, then $LG\text{–}\dim R = 1 + \max\{SP\text{–}\dim(I_{j-1}, I_j) : j = 1, \ldots, n\}$.

Proposition. Suppose $B, C \in {}_RM$. Then

$$SP\text{–}\dim(B \oplus C) = \max\{SP\text{–}\dim B, SP\text{–}\dim C\}$$

Corollary. If $LG\text{–}\dim R > 0$, and if $R = I_1 \oplus \cdots \oplus I_n$ is a direct sum of left ideals, then $LG\text{–}\dim R = 1 + \max\{SP\text{–}\dim I_j : j = 1, \ldots, n\}$.

Proposition. Suppose $\phi : R \to \hat{R}$ is a surjective ring homomorphism and suppose $\widehat{R}$ is R-projective. Then $P\text{–}\dim {}_R\widehat{B} = P\text{–}\dim {}_{\widehat{R}}\widehat{B}$ for all $\widehat{B} \in {}_{\widehat{R}}M$.

Theorem. For any ring R, and any positive integer n, $LG\text{–}\dim R = LG\text{–}\dim M_n(R)$.

Proposition. Suppose a is central in R, and a is neither a unit nor a zero divisor. Set $\widehat{R} = R/Ra$. Suppose $\widehat{B}$ is a nonzero $\widehat{R}$-module with finite projective dimension as an $\widehat{R}$-module. Then

$$P\text{–}\dim_R \widehat{B} = 1 + P\text{–}\dim_{\widehat{R}} \widehat{B}$$

Corollary. Suppose a is central in R, and a is neither a unit nor a zero divisor. Set $\widehat{R} = R/Ra$, and suppose $LG\text{–}\dim \widehat{R} < \infty$. Then $LG\text{–}\dim R \ge 1 + LG\text{–}\dim \widehat{R}$.

Corollary. If $LG\text{–}\dim R < \infty$, then $LG\text{–}\dim R[x] \ge 1 + LG\text{–}\dim R$.

Given $B \in {}_RM$, set $B[x] = R[x] \otimes_R B$.

Proposition. For all $B \in {}_RM$, $P\text{–}\dim_{R[x]} B[x] = P\text{–}\dim_R B$.

Theorem. If R is any ring, then $LG\text{–}\dim R[x] = 1 + LG\text{–}\dim R$.

Corollary. If K is a field, then $LG\text{–}\dim K[x_1, \ldots, x_n] = n$.

Call a multiplicatively closed subset $S \subset R$ *admissible* if $0 \notin S$ and $1 \in S$.

Proposition. Suppose R is a commutative ring, and S is an admissible subset of R. Suppose $B \in {}_R M$. Then

1. $S^{-1}B \cong S^{-1}R \otimes_R B$.

2. $S^{-1}R$ is flat as an R-module.

Call a functor *strongly additive* if $F(\oplus_I B_i) \cong \oplus_I F(B_i)$ for any indexed family in ${}_R M$.

Corollary. Suppose R is a commutative ring and S is an admissible subset of R. Then the map $B \mapsto S^{-1}B$ is an exact, strongly additive covariant functor.