Theorem (Binomial Theorem)**.** For all real numbers a and b, and for all non-negative integers n, $(a+b)^n = \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} b^k$.

Lemma. For non-negative integers n and k, $n \leq k$, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

Proof. By straightforward computation,

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k(k-1)!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k)!(n-k+1)} \\
&= \frac{n!(n+1)}{k!(n-k+1)}! \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \binom{n+1}{k}. \quad \square
\end{aligned}$$

Proof of Binomial Theorem. We do the proof by mathematical induction. The theorem is clearly true for $n = 0$ and $n = 1$. Suppose the theorem is true for some positive integer n. Then,

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\left(\sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} b^k\right) \\
&= \left(\sum_{k=0}^{k=n} \binom{n}{k} a^{n-k+1} b^k\right) + \left(\sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} b^{k+1}\right) \\
&= \binom{n}{0} a^{n+1} + \left(\sum_{k=1}^{k=n} \left(\binom{n}{k} + \binom{n}{k-1}\right) a^{(n+1)-k} b^k\right) + \binom{n}{n} b^{n+1} \\
&= \binom{n+1}{0} a^{n+1} + \sum_{k=1}^{k=n} \binom{n+1}{k} a^{(n+1)-k} b^k + \binom{n+1}{n+1} b^{n+1} \\
&= \sum_{k=0}^{k=n+1} \binom{n+1}{k} a^{(n+1)-k} b^k
\end{aligned}$$

So the theorem is true for the integer $n+1$. Thus we conclude that the theorem is true for all non-negative integers n. $\square$