Notes from Real and Complex Analysis

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Contents

Part I

Real Analysis

Chapter 1

Elementary Real Analysis

1.1 Continuous Functions and Derivatives

Definition 1.1.1. Let $f : D \to \mathbb{R}$ be a function. We say that f is continuous at $x \in D$ if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that $\forall\, y \in D$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. We say that f is continuous on D if f is continuous at each $x \in D$.

Definition 1.1.2. Let $D \subset \mathbb{R}$ be an open set, and let $f : D \to \mathbb{R}$ be a function. We say that f is differentiable at $x_0 \in D$ if the limit $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite. We say that f is differentiable on D if f is differentiable at each $x \in D$.

Definition 1.1.3. We call a set $A \subset \mathbb{R}$ compact if every open cover of A has a finite subcover.

Theorem 1.1.4 (Bolzano-Weierstrass Theorem)**.** Every infinite subset of a compact set D contains a limit point in D.

Proof. Let D be a compact space, and let $S \subset D$ be a set having no limit points. For each $x \in D$, there exists an open set O_x such that $O_x \cap S = \emptyset$ if $x \notin S$, and $O_x \cap S = \{x\}$ if $x \in S$. Then the collection of open sets O_x is an open cover of D, hence has a finite subcover. But each open set contains at most one point of S. Conclude that S is finite. □

Lemma 1.1.5. Let $\{A_i\}_1^\infty$ be a collection of nonempty, closed and bounded subsets of $\mathbb{R}$ such that $A_{i+1} \subset A_i$ for all $i \geq 1$. Then $\cap_1^\infty A_i \neq \emptyset$.

Proof. For each $i \geq$, let $a_i \in A_i$. Then $\{a_i\}_1^\infty$ is an infinte bounded subset of the compact set A_1, hence contains a limit point $a \in A_1$ by the Bolzano-Weierstrass Theorem. Let $\epsilon > 0$. Then for k sufficiently large, $a_k \in (a - \epsilon, a + \epsilon)$. Since the sets A_i are nested, we conclude that $(a - \epsilon, a + \epsilon)$ contains a point in each A_i. Then either $a \in A_i$ for all $i \geq 1$, or a is a limit point of each A_i. But the A_i are closed, hence contain all of their limit points. Conclude $a \in \cap_1^\infty A_i$. □

Corollary 1.1.6. Let $\{A_1, A_2, \ldots\}$ be a countable collection of closed bounded subsets of $\mathbb{R}$ such that $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$. If $\cap_{i=1}^{\infty} A_i = \emptyset$, then $\cap_1^N A_i = \emptyset$ for some $N \in \mathbb{N}$.

Theorem 1.1.7 (Heine-Borel Theorem)**.** A subset $D \subset \mathbb{R}$ is compact iff it is closed and bounded.

Proof. Let $D \subset \mathbb{R}$ be a compact set. If D is unbounded, then $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover of D having no finite subcover. If D is not closed, then there exists some limit point a of D not contained in D, in which case $\{(-\infty, a - \frac{1}{n}) \cup (a + \frac{1}{n}, \infty) : n \in \mathbb{N}\}$ is an open cover of D having no finite subcover. Conclude that D is closed and bounded.

Let D be a closed and bounded subset of $\mathbb{R}$, and let $\{I_n\}_1^{\infty}$ be an open cover of D. Let $J_n = \cup_1^n I_i$. Then $\{J_n\}_1^{\infty}$ is an open cover of D. Let $K_n = D \backslash J_n$. Then $\{K_n\}_1^{\infty}$ is a decreasing sequence closed sets. Now $D \backslash (\cup_1^{\infty} J_n) = \emptyset$, so $\emptyset = \cap_1^{\infty} K_n$. Then by the above corollary, there exists an $N \in \mathbb{N}$ such that $\cap_1^N K_n = \emptyset$. Then $D \subset \cup_1^N J_n = \cup_1^N I_n$. Conclude that D is compact. $\square$

Theorem 1.1.8 (Extreme Value Theorem)**.** A continuous function on a comapact set D attains its supremum and infimum on D.

Proof. Let f be a continuous function on the compact domain D. Let $M = \sup\{f(x) : x \in D\}$, and let $\{x_n\}_1^{\infty} \subset D$ such that $\lim_{n\to\infty} f(x_n) = M$. By the Bolzano-Weierstrass Theorem, $\{x_n\}_1^{\infty}$ must have a limit point $x \in D$. Then $f(x) = f(\lim x_n) = \lim f(x_n) = M$. By a similar argument, there exists $y \in D$ such that $f(y) = \inf\{f(x) : x \in D\}$. $\square$

Theorem 1.1.9. If f is a continuous function on $[a, b]$ and if $f(a), f(b)$ are of opposite signs, then there is a number $c \in [a, b]$ such that $f(c) = 0$.

Proof. Assume $f(a) > 0, f(b) < 0$ (otherwise consider the function $-f(x)$). Let $A = \{t \in [a, b] : f(x) > 0 \text{ if } x \in [a, t]\}$. Then $A \neq \emptyset$. Let $c = \sup A$. Then $a < c < b$. Moreover, we must have $f(c) = 0$ (or else c is not the sup of A). $\square$

Corollary 1.1.10 (Intermediate Value Theorem)**.** If f is continuous on $[a, b]$ and d is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = d$.

Lemma 1.1.11. Let $f : (a, b) \to \mathbb{R}$ have a relative extrema at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.

Corollary 1.1.12 (Rolle's Theorem)**.** Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Either f is constant on $[a, b]$, in which case $f'(c) = 0$ for all $c \in (a, b)$, or else f has a relative extrema at some $c \in (a, b)$, in which case $f'(c) = 0$. $\square$

Theorem 1.1.13 (Mean Value Theorem)**.** Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof. Apply Rolle's Theorem to the function $g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right] x$. □

Definition 1.1.14. We call a function $f : A \to \mathbb{R}$ uniformly continuous if for all $\epsilon > 0$, $\exists\, \delta > 0$ such that if $x, y \in A$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Theorem 1.1.15. A continuous function on a compact set is uniformly continuous.

Proof. Let f be a continuous function on the compact set D. Let $\epsilon > 0$. Then for each $x \in D$, $\exists\, \delta_x > 0$ such that if $y \in D$ and $|x - y| < \delta_x$, then $|f(x) - f(y)| < \epsilon$. Since D is compact, we can choose $x_1, x_2, \ldots, x_n \in D$ such that $\{(x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2) : 1 \leq i \leq n\}$ covers D. Let $\delta = \min\{\delta_{x_i}/2 : 1 \leq i \leq n\}$. So $\delta > 0$. Let $x, y \in D$, and suppose $|x - y| < \delta$. We have $x \in (x_j - \delta_{x_j}/2, x_j + \delta_{x_j}/2)$ for some $1 \leq j \leq n$. Then $|x - y| < \delta \leq \delta_{x_j}/2 \Rightarrow |x_j - y| < \delta_{x_j}$. Then $|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| \leq 2\epsilon$. Conclude that f is uniformly continuous. □

1.2 Infinite Series and Sequences

Definition 1.2.1. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of real numbers. We say that the series $\sum_{i=1}^{\infty} a_i$ converges if the limit $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists and is finite. We say that the series converges absolutely if the series $\sum_{i=1}^{\infty} |a_i|$ converges. We say that the series $\sum_{i=1}^{\infty} a_i$ converges conditionally if it converges but does not converge absolutely.

Example 1.2.2.

(a) The geometric series $\sum_{n=1}^{\infty} a^n$ converges to $\frac{1}{1-a}$ if $|a| < 1$, and diverges if $|a| \geq 1$.

(b) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

(c) The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Remark 1.2.3 (Tests for Convergence)**.** Let $\sum_1^{\infty} a_i$, $\sum_1^{\infty} b_1$ be series of non-negative terms.

(a) (Comparison Test) Assume $a_i \leq b_i$ for all $i \geq 1$. If $\sum_1^{\infty} b_i$ converges, then so does $\sum_1^{i} nfty a_i$. If $\sum_1^{\infty} a_i$ diverges, then so does $\sum_1^{\infty} b_i$.

(b) (Ratio Test) Assume $a_i \neq 0$ for all $i \in \mathbb{N}$. The series converges if $R = \limsup \frac{a_{i+1}}{a_i} < 1$, and diverges if $r = \liminf \frac{a_{i+1}}{a_i} > 1$. The test is inconclusive if $r \leq 1 \leq R$.

(c) (Root Test) Let $\rho = \limsup \sqrt[n]{a_i}$. The series converges if $\rho < 1$, and diverges if $\rho > 1$. The test is inconclusive if $\rho = 1$.

(d) (Integral Test) Assume $a_1 \geq a_2 \geq a_3 \geq \cdots$. Let $f(x)$ be a nonincreasing continuous function on $(0, \infty)$ such that $f(i) = a_i$ for all $i \in \mathbb{N}$. Then the series $\sum_1^\infty$ converges iff the improper integral $\int_1^\infty f(x)\,dx$ converges. If the improper integral converges, then it is an upper bound for the value of the series.

Definition 1.2.4. Let $\{f_n(x)\}_{n=1}^\infty$ be a sequence of functions on a domain D. Suppose that for each $x_0 \in D$, $\{f_n(x_0)\}_1^\infty$ is a convergence sequence. Defined a function f on D by $f(x) := \lim_{n\to\infty} f_n(x)$. We say that the sequence $\{f_n(x)\}_1^\infty$ converges to f uniformly on D if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that if $m \geq N$, $|f_m(x) - f(x)| < \epsilon$ for all $x \in D$.

Definition 1.2.5. Say a sequence of functions $\{f_n(x)\}_1^\infty$ on a domain D is uniformly Cauchy on D if for all $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that if $i, j \geq N$, then $|f_i(x) - f_j(x)| < \epsilon$ for all $x \in D$.

Theorem 1.2.6 (Cauchy Criterion for Uniform Convergence)**.** Let $\{f_n(x)\}_1^\infty$ be a sequence of functions on a domain D. Then $\{f_n(x)\}_1^\infty$ converges uniformly to a function f on D iff $\{f_n(x)\}_1^\infty$ is uniformly Cauchy on D.

Proof. Suppose $\{f_n(x)\}_1^\infty$ converges uniformly to f on D. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in D$. Then if $m, n \geq N$, $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \epsilon$ for all $x \in D$.

Suppose $\{f_n(x)\}_1^\infty$ is uniformly Cauchy on D. For each $x_0 \in D$, the sequence $\{f_n(x_0)\}_1^\infty$ is Cauchy, hence converges, so we may define $f(x) = \lim_{n\to\infty} f_n(x)$. Let $N \in \mathbb{N}$ such that if $m, n \geq N$, $|f_n(x) - f_m(x)| < \epsilon/2$ for all $x \in D$. Let $x \in D$, and let $n \geq N$. There exists $m \geq n$ (depending on x) satisfying $|f_m(x) - f(x)| < \epsilon/2$. Then $|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \epsilon$. Conclude that $\{f_n(x)\}_1^\infty$ converges uniformly to f. □

Definition 1.2.7. For each $n \in \mathbb{N}$, let $f_n(x)$ be a real-valued function. We say that the series $\sum_1^\infty f_n(x)$ converges on a domain D if for each fixed $x_0 \in D$, the series $\sum_1^\infty f_n(x_0)$ converges. We say the series converges uniformly on D if the sequence of partial sums $S_m(x) = \sum_1^m f_n(x)$ converges uniformly on D.

Theorem 1.2.8 (Weierstrass M-test)**.** Let $\{f_n(x)\}_{n=1}^\infty$ be a sequence of functions on a domain D such that for each $n \in N$, $\exists\, M_n \geq 0$ satisfying $|f_n(x)| \leq M_n$ for all $x \in D$, and $\sum_1^\infty M_n < \infty$. Then the series $\sum_1^\infty f_n(x)$ converges uniformly on D.

Proof. Let $\epsilon > 0$. Since $\sum_1^\infty M_n < \infty$, there exists $N \in \mathbb{N}$ such that if $i, j \geq N$, $\sum_i^j M_n < \epsilon$. Now if $i, j \geq N$, $\left|\sum_i^j f_n(x)\right| \leq \sum_i^j |f_n(x)| \leq \sum_i^j M_n < \epsilon$ for all $x \in D$. Then by the Cauchy Criterion for Uniform Convergence, $\{f_n(x)\}_1^\infty$ converges uniformly on D. □

Definition 1.2.9. Let I be an open interval, $a \in I$, and suppose $f \in C^n(I)$. Then the Taylor polynomial of degree n generated by f at the point $x = a$, denoted $T_n(f;a)(x)$ is defined to be the finite series $T_n(f;a)(x) = \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(x-a)^k$. If $f \in C^\infty(I)$, then the Taylor Series of $f(x)$ about $x = a$ is defined to be the series $\sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$.

Definition 1.2.10. Let I be an open interval, $c \in I$, and suppose $f \in C^n(I)$. We define the n-th Taylor remainder function for the Taylor polynomial of degree n generated by f at $x = a$, denoted $R_n(x)$, by $R_n(x) = f(x) - T_n(f;a)(x)$.

Theorem 1.2.11. Let $f \in C^{n+1}(I)$, where I is an open interval containing $[a,b]$. Then there is a point $c \in (a,b)$ such that $f(x) - T_n(f;a)(b) = R_n(b) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$.

1.3 Integration

Definition 1.3.1. A partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a,b]$ is a finite subset of $[a,b]$ such that $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$.

Definition 1.3.2. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. We define the upper and lower Riemann sums, respectively, of f on $[a,b]$ with respect to the partition P by $\overline{S}(f;P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$, $\underline{S}(f;P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$, where $m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. We define $\overline{S}(f)$ be to the the infimum of $\overline{S}(f;P)$ taken over all partitions P of $[a,b]$, and we define $\underline{S}(f)$ to be the supremum of $\underline{S}(f;P)$ taken over all partitions P of $[a,b]$.

Definition 1.3.3. Let f be a bounded function on the interval $[a,b]$. We say that f is Riemann integrable on $[a,b]$ if $\overline{S}(f) = \underline{S}(f)$. We denote this common value by $\int_a^b f(x)\,dx$.

Theorem 1.3.4. Every continuous real-valued function on the finite interval $[a,b]$ is Riemann integrable on $[a,b]$.

Theorem 1.3.5 (Riemann-Lebesgue Theorem)**.** A bounded continuous function f on the finite interval $[a,b]$ is Riemann integrable on $[a,b]$ iff the set of discontinuities of f on $[a,b]$ has (Lebesgue) measure zero.

Theorem 1.3.6 (Mean Value Theorem for Integrals)**.** Let f be continuous on the finite interval $[a,b]$. Then $\exists\, c \in [a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Proof. Let $m = \inf\{f(x) : x \in [a,b]\}$, $M = \sup\{f(x) : x \in [a,b]\}$. Then

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

which implies $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$, i.e, $m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$. Then by the Intermediate Value Theorem, there exists $c \in [a,b]$ such that $f(c)(b-a) = \int_a^b f(x)\,dx$. □

Theorem 1.3.7 (Fundamental Theorem of Calculus)**.** Let f be a Riemann integrable function on the finite interval $[a,b]$, and let F be a function on $[a,b]$ such that F is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$ on (a,b). Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Theorem 1.3.8. Let $\{f_n\}$ be a sequence of functions that are differentiable on an open interval containing the finite interval $[a,b]$. If $\sum f_n(x_0)$ exists for some $x_0 \in [a,b]$, and if $\sum f_n'(x)$ converges uniformly on $[a,b]$, then $\sum f_n(x)$ converges uniformly on $[a,b]$, and $\sum f_n'(x) = \frac{d}{dx}\left(\sum f_n(x)\right)$.

Theorem 1.3.9 (Weierstrass Approximation Theorem)**.** Any continuous real-valued function on a bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

Chapter 2

Measure Theory

2.1 Measures

Definition 2.1.1. Let X be a nonempty set. An algebra of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements. A σ-algebra of sets on X is an algebra which is closed under countable unions.

Definition 2.1.2. Define a monotone class on X to be a subset $\mathcal{C}$ of $\mathcal{P}(X)$ that is closed under countable increasing unions and countable decreasing intersections.

Lemma 2.1.3 (Monotone Class Lemma)**.** If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.

Definition 2.1.4. Let X be a nonempty set and $\mathcal{M}$ a σ-algebra on X. A measure on X with domain $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu(\emptyset) = 0$, and if $\{E_i\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\cup_1^\infty E_i) = \sum_1^\infty \mu(E_i)$. Call $(X, \mathcal{M}, \mu)$ a measure space if X is a set, $\mathcal{M}$ is a σ-algebra on X, and μ is a measure on X with domain $\mathcal{M}$.

Definition 2.1.5. Let X be a set, and let μ be a measure defined on some subset of $\mathcal{P}(X)$. A set $A \in \mathcal{P}(X)$ is called μ-measurable if A is in the domain of μ.

Definition 2.1.6. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $\mu(X) < \infty$, say that μ is finite. If $X = \cup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$, say that μ is σ-finite.

Definition 2.1.7. A measure is called complete if whenever $A \in \mathcal{M}$ and $\mu(A) = 0$, then $B \in \mathcal{M}$ for all $B \subseteq A$.

Theorem 2.1.8. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \le \mu(F)$.

(b) (Subadditivity) If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\cup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.

(c) (Continuity from below) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_i \subset E_{i+1}$, then $\mu(\cup_1^\infty E_j) = \lim\limits_{j\to\infty} \mu(E_j)$.

(d) (Continuity from above) If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_{i+1} \subset E_i$, and $\mu(E_1) < \infty$, then $\mu(\cap_1^\infty E_j) = \lim\limits_{j\to\infty} \mu(E_j)$.

Proof.

(a) Have $\mu(F) = \mu(E \cup (F\backslash E)) = \mu(E) + \mu(F\backslash E)$, so $\mu(E) \leq \mu(F)$.

(b) Let $F_j = E_j \backslash \left(\cup_1^{j-1} E_i\right)$. Then $\mu(\cup_1^\infty E_j) = \mu(\cup_1^\infty F_j) = \sum_1^\infty \mu(F_j) \leq \sum_1^\infty \mu(E_j)$.

(c) Let $F_j = E_j \backslash E_{j-1}$. Then $\mu(\cup_1^\infty E_j) = \mu(\cup_1^\infty F_j) = \sum_1^\infty \mu(F_j) = \lim_{n\to\infty} \sum_1^n \mu(F_j)$. But $\sum_1^n \mu(F_j) = \mu(\cup_1^n F_j) = \mu(E_n)$. Then $\mu(\cup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

(d) Let $F_j = E_1 \backslash E_j$. Then $F_j \subset F_{j+1}$, and $\cup_1^\infty F_j = E_1 \backslash \left(\cap_1^\infty E_j\right)$. Then $\mu(E_1) - \mu(\cap_1^\infty E_j) = \mu(E_1 \backslash (\cap_1^\infty E_j)) = \mu(\cup_1^\infty F_j) = \lim_{j\to\infty} \mu(F_j) = \lim_{j\to\infty} \mu(E_1 \backslash E_j) = \lim_{j\to\infty} \mu(E_1) - \mu(E_j)$. Conclude $\mu(\cap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$. $\square$

Definition 2.1.9. An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ satisfying: (i) $\mu^*(\emptyset) = 0$, (ii) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$, (iii) $\mu^*(\cup_1^\infty A_i) \leq \sum_1^\infty \mu^*(A_i)$.

Definition 2.1.10. Let μ^* be an outer measure on a set X. Call a set $A \subseteq X$ μ^*-measurable if for all $E \subseteq X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Remark 2.1.11. In the definition of μ^* measurability above, we have $E \subset (E \cap A) \cup (E \cap A^c)$, so the inequality $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ always holds.

Theorem 2.1.12 (Carathéodory's Theorem)**.** If μ^* is an outer measure on a nonempty set X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Definition 2.1.13. If $\mathcal{A}$ is an algebra on a nonempty set X, a premeasure on $\mathcal{A}$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ satisfying: (i) $\mu_0(\emptyset) = 0$, (ii) If $\{A_i\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\cup_1^\infty A_i \in \mathcal{A}$, then $\mu_0(\cup_1^\infty A_i) = \sum_1^\infty \mu_0(A_i)$.

Remark 2.1.14. Given a nonempty set X, an algebra $\mathcal{A}$ on X, and a premeasure $\mu_0 : \mathcal{A} \to [0, \infty]$, μ_0 induces an an outer measure μ^* on X given by $\mu^*(E) = \inf\{\sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_1^\infty A_j\}$.

Theorem 2.1.15. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. Let μ^* be the outer measure induced on X by μ_0. Then $\mu = \mu^*|_{\mathcal{M}}$ is a measure, and $\mu|_{\mathcal{A}} = \mu_0$. If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

2.2 Borel Measures

Definition 2.2.1. We define the Borel σ-algebra on $\mathbb{R}$, denoted $\mathfrak{B}_{\mathbb{R}}$ to be the σ-algebra on $\mathbb{R}$ generated by all open subsets of $\mathbb{R}$. Any measure μ with domain $\mathfrak{B}_{\mathbb{R}}$ is called a Borel measure on $\mathbb{R}$.

Definition 2.2.2. Define an h-interval to be a set of the form $(a, b]$, where $-\infty \leq a < b < \infty$.

Proposition 2.2.3. Let $\mathcal{A}$ denote the algebra consisting of all finite disjoint unions of h-intervals. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Given a disjoint collection $\{(a_j, b_j]\}_1^n$ of h-intervals, define $\mu_0(\cup_1^n (a_j, b_j]) = \sum_1^n [F(b_j) - F(a_j)]$, and let $\mu_0(\emptyset) = 0$. Then μ_0 is a premeasure on $\mathcal{A}$.

Theorem 2.2.4. If $F : \mathbb{R} \to \mathbb{R}$ is an increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0, x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x, 0]) & x < 0 \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

Definition 2.2.5. Let m denote the complete Borel measure μ_F on $\mathbb{R}$ associated to the function $F(x) = x$. We call this measure Lebesgue measure (or Lebesgue-Stieltjes measure) on $\mathbb{R}$, and denote the domain of m by $\mathcal{L}$. Say a set $X \subset \mathbb{R}$ is Lebesgue measurable if $X \in \mathcal{L}$.

Definition 2.2.6. Let μ be a Borel measure on $\mathbb{R}$. The measure μ is called outer regular if $\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ open}\}$ for all $E \in \mathfrak{B}_{\mathbb{R}}$. The measure μ is called inner regular if $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ for all compact sets $K \in \mathfrak{B}_{\mathbb{R}}$. The measure μ is called regular if it is both outer and inner regular.

Theorem 2.2.7. Any complete Lebesgue-Stieltjes measure μ on $\mathbb{R}$ is regular.

Corollary 2.2.8. Given a complete Lebesgue-Stieltjes measure μ on $\mathbb{R}$ with domain $\mathcal{M}$, and a set $E \in \mathcal{M}$ with $\mu(E) < \infty$, for every $\epsilon > 0$, there exists a set A that is a finite union of disjoint open intervals such that $\mu((E \backslash A) \cup (A \backslash E)) < \epsilon$.

Theorem 2.2.9. Given $E \in \mathcal{L}$ and $r \in \mathbb{R}$, $E + r \in \mathcal{L}$, $rE \in \mathcal{L}$, $m(E + r) = m(E)$, and $m(rE) = |r|m(E)$. In particular, Lebesuge measure is translation invariant.

Proof. The result is true for finite unions of intervals, hence is true for all $E \in \mathcal{B}_{\mathbb{R}}$. If $E \in \mathcal{B}_{\mathbb{R}}$ and $m(E) = 0$, then any translation of dilation of E is also a null set. Then the class of sets of Lebesgue measure zero is also translation and dilation invariant. Hence the result is true for all $E \in \mathcal{L}$, because each element of $\mathcal{L}$ can be expressed as the union of a Borel set and a Lebesgue null set. □

2.3 Measurable Functions

Definition 2.3.1. If $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ are measure spaces, a map $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Proposition 2.3.2. The sum and product of measurable functions is measurable.

Proposition 2.3.3. If $\{f_j(x)\}_1^\infty$ is a sequence of real-valued measurable functions, then $\sup_j f_j(x)$ and $\inf_j f_j(x)$ are measurable functions. In particular, if f, g are real-valued measurable functions, then so are $\max(f, g)$ and $\min(f, g)$.

Corollary 2.3.4. If $\{f_j\}_1^\infty$ is a sequence of real or complex-valued functions converging to f, then f is measurable.

Proposition 2.3.5. A complex-valued function $f : X \to \mathbb{C}$ is measurable iff $\mathrm{Re}\,(f)$ and $\mathrm{Im}\,(f)$ are measurable.

Definition 2.3.6. Given a measure space $(X, \mathcal{M}, \mu)$, a simple functon on X is a finite $\mathbb{C}$-linear combination of characteristic functions of sets in $\mathcal{M}$.

Chapter 3

The Lebesgue Integral

3.1 Convergence Theorems

Definition 3.1.1. Given a measure space $(X, \mathcal{M}, \mu)$, define L^+ = space of measurable functions from X to $[0, \infty]$.

Definition 3.1.2. If $\phi \in L^+$, $\phi = \sum_1^n a_j \chi_{E_j}$, we define the integral of ϕ with respect to the measure μ by $\int \phi \, d\mu = \sum_1^n a_j \mu(E_j)$.

Theorem 3.1.3 (Monotone Convergence Theorem)**.** Given a sequence $\{f_n\}_1^\infty$ of measurable functions on a measure space $(X, \mathcal{M}, \mu)$ with $0 \leq f_n \leq f_{n+1}$, then $\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$.

Proof. Since $f_n(t) \leq f(t) := \lim_{n\to\infty} f_n(t)$ for all $n \in N$, $\int f_n(t) \, d\mu \leq \int f(t) \, d\mu$. Then $\lim_{n\to\infty} \int f_n(t) \, d\mu \leq \int f(t) \, d\mu$. Let $\alpha \in (0, 1)$, and let ϕ be a simple function with $0 \leq \phi \leq f$. Let $E_n = \{x : f_n(x) \geq \alpha\phi(x)\}$. Then $E_n \subseteq E_{n+1}$, and $\cup_1^\infty E_n = X$. Now $f_n \chi_{E_n} \leq f_n$, and $\int f_n \geq \int_{E_n} f_n \geq \int_{E_n} \alpha\phi$. Then $\lim_{n\to\infty} \int f_n \geq \lim_{n\to\infty} \int_{E_n} \alpha\phi$. Say $\phi(x) = \sum_1^j c_i \chi_{A_i}(x)$. Then $\lim_{n\to\infty} \int_{E_n} \alpha\phi(x) \, d\mu = \lim_{n\to\infty} \sum_1^j \alpha c_i \chi_{A_i \cap E_n}(x) = \sum_1^j \alpha c_i \chi_{A_i} = \int \alpha\phi$. This is true for all $\alpha < 1$ so it is true for $\alpha = 1$. So $\lim_{n\to\infty} \int f_n \geq \int \phi$. Take the supremum over all simple functions $\phi \leq f$, and we have $\lim_{n\to\infty} f_n \geq \int f$. $\square$

Lemma 3.1.4 (Fatou's Lemma)**.** If $\{f_n\}_1^\infty \subset L^+$, then $\int \liminf f_n \leq \liminf \int f_n$.

Proof. Let $h_n(x) = \inf_{k \geq n} f_k(x)$. Then $\{h_n\}_1^\infty$ is an increasing sequence in L^+, so by the Monotone Convergence Theorem, $\int \liminf f_n = \int \lim_{n\to\infty} h_n = \lim_{n\to\infty} \int h_n$. Now $h_n(x) \leq f_k(x)$ for all $k \geq n \Rightarrow \int h_n \leq \int f_k(x)$ for all $k \geq n$, i.e., $\int h_n \leq \inf_{k \geq n} \int f_k$. Then $\lim_{n\to\infty} \int h_n \leq \lim_{n\to\infty} \inf_{k \geq n} \int f_k = \liminf \int f_n$. $\square$

Definition 3.1.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a real or complex-valued functon on X. Say f is integrable if $\int |f| < \infty$. If f is real-valued and integrable, define $\int f = \int f^+ - \int f^-$, where $f^+ = \max(f, 0)$, and $f^- = \min(f, 0)$. If f is complex-valued and integrable, define $\int f = \int \mathrm{Re}\,(f) + i \int \mathrm{Im}\,(f)$. Denote the space of complex-valued integrable functions on X by $L^1(X)$.

Proposition 3.1.6. If $f \in L^1$, then $\left|\int f\right| \leq \int |f|$.

Proposition 3.1.7. If $f, g \in L^1$, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f - g| = 0$ iff $f = g$ almost everywhere.

Remark 3.1.8. We can define a metric $\|\cdot\|_1$ on L^1 be $\|f\|_1 = \int |f|$.

Theorem 3.1.9 (Dominated Convergence Theorem)**.** Let $\{f_n\} \subset L^1$ such that $f_n \to f$ a.e., and there exists $g \in L^+$ such that $|f_n| \leq g$ a.e. for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Proof. Since f is the limit of measurable functions, it is measurable. Also, $|f_n| \leq g$ for all $n \in \mathbb{N} \Rightarrow f = \lim |f_n| \leq g$, so $f \in L^1$. By taking real and imaginary parts it suffices to assume that f_n and f are real-valued, in which case we have $g + f_n \geq 0$ and $g - f_n \geq 0$ a.e. Then by Fatou's Lemma,

$$\begin{aligned} \int g + \int f &\leq \liminf \int (g + f_n) = \int g + \liminf \int f_n \\ \int g - \int f &\leq \liminf \int (g - f_n) = \int g + \limsup \int f_n \end{aligned}$$

Then $\limsup \int f_n \leq \int f \leq \liminf \int f_n$, hence $\lim \int f_n$ exists and equals $\int f$. □

Theorem 3.1.10. The integrable simple functions $\phi = \sum a_j \chi_{E_j}$ are dense in the $L^1(\mathbb{R}, \mu)$ metric. If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, the sets E_j in the definition of ϕ can be taken to be finite unions of disjoint open intervals (i.e., the "really simple" functions are dense in $L^1(\mathbb{R}, \mu)$). In this case also, the infinitely differentiable continuous functions of compact support are dense in the $L^1(\mathbb{R}, \mu)$ metric.

3.2 n-Dimensional Lebesgue Integration

Definition 3.2.1. Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be measure spaces. Define $\mathcal{M} \otimes \mathcal{N}$ to be the σ-algebra generated by all sets of the form $A \times B$, where $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

Definition 3.2.2. Given $E \in \mathcal{M} \otimes \mathcal{N}$, define the x-section E_x of E by $E_x = \{y : (x, y) \in E\}$, and define the y-section E^y of E by $E^y = \{x : (x, y) \in E\}$. Given an $\mathcal{M} \otimes \mathcal{N}$ measurable function f, define the x-section f_x and the y-section f^y of f by $f_x(y) = f^y(x) = f(x, y)$.

Proposition 3.2.3. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$ for all $x \in X$ and $y \in Y$. If f is $\mathcal{M} \otimes \mathcal{N}$ measurable, then f_x is $\mathcal{N}$-measurable, and f^y is $\mathcal{M}$-measurable for all $x \in X$ and $y \in Y$.

Definition 3.2.4. Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be measure spaces. Define $\mu \times \nu$, the product measure of μ and ν, to be the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ for all rectangles $A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$.

Theorem 3.2.5. Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E_y)$ are measurable on X and Y, respectively, and $\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$.

Theorem 3.2.6 (Fubini-Tonelli Theorem)**.** Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let f be a $\mathcal{M} \otimes \mathcal{N}$ measurable function. Then $\int |f|\, d(\mu \times \nu) = \iint |f|\, d\nu\, d\mu = \iint |f|\, d\mu\, d\nu$, where if one quanitity is infinite, all three are infinite. If one of the preceding quantities is finite, then $\int f\, d(\mu \times \nu) = \iint f\, d\nu\, d\mu = \iint f\, d\mu\, d\nu$.

Theorem 3.2.7. Lebesgue measure is translation and rotation invariant.

Theorem 3.2.8. Let $T \in GL_n(\mathbb{R})$. (a) If f is Lebesgue measurable on $\mathbb{R}^n$, so is $f \circ T$. If $f \geq 0$ or $f \in L^1(m)$, then $\int f(x)\, dx = |\det T| \int f \circ T(x)\, dx$. (b) If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T| m(E)$.

Theorem 3.2.9. Suppose Ω is an open set in $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. (a) If f is Lebesgue measurable on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then $\int_{G(\Omega)} f(x)\, dx = \int_\Omega f \circ G(x) |\det D_x G|\, dx$. (b) If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E |\det D_x G|\, dx$.

Proposition 3.2.10. If f is measurable on $\mathbb{R}^n$, nonnegative or integrable, such that $f(x) = g(|x|)$ for some function g on $(0, \infty)$, then $\int f(x)\, dx = \sigma(S^{n-1}) \int_0^\infty g(r) r^{n-1}\, dr$.

Chapter 4

L^p Spaces

4.1 Basic Results

Definition 4.1.1. Given a measure space $(X, \mathcal{M}, \mu)$, a measurable function f on X, and $0 < p < \infty$, define $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$. Define $L^p(X, \mathcal{M}, \mu)$ to be the set of all complex-valued measurable functions on X such that $\|f\|_p < \infty$.

Remark 4.1.2. For $p \geq 1$, $\| \cdot \|_p$ is a norm on L^p.

Theorem 4.1.3 (Hölder's Inequality)**.** Suppose $1 \leq p \leq \infty$ and $p^{-1} + q^{-1} = 1$. If f, g are measurable functions on X, $f \in L^p$, $g \in L^q$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$, with equality iff $\alpha|f|^p = \beta|g|^q$ for some constants α, β with $\alpha\beta \neq 0$.

Theorem 4.1.4 (Minkowski's Inequality)**.** If $1 \leq p < \infty$ and $f, g \in L^p$, then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Theorem 4.1.5. For $1 \leq p < \infty$, L^p is a Banach space, i.e., the set of all complex-valued measurable functions f on X with $\|f\|_p < \infty$ is a complete complex vector space with norm $\| \cdot \|_p$.

Proposition 4.1.6. For $1 \leq p < \infty$, the simple funtions $\phi = \sum_1^n a_j \chi_{E_j}$, with $\mu(E_j) < \infty$ for all j, are dense in L^p. If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, then the infinitely differentiable functions of compact support are dense in L^p.

Definition 4.1.7. If f is measurable on X, define the essential supremum of f, denoted $\|f\|_\infty$, by $\|f\|_\infty = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}$. Define L^∞ to be the set of all complex-valued measurable functions f on X such that $\|f\|_\infty < \infty$.

Theorem 4.1.8. (a) If f, g are measurable functions on X, then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$, equality above holds iff $|g| = \|g\|_\infty$ almost everywhere on the set where $f(x) \neq 0$. (b) $\|\cdot\|_\infty$ is a norm on L^∞. (c) L^∞ is a Banach space. (d) The simple functions are dense in L^∞.

Proposition 4.1.9. Suppose p and q are conjugate exponents, and $1 \leq q < \infty$. If $g \in L^q$, then $\|g\|_q = \sup\{|\int fg| : \|f\|_p = 1\}$.

Theorem 4.1.10. Let p and q be conjugate exponents, μ a σ-finite measure, $1 \leq p < \infty$. Let ϕ be a bounded linear functional on L^p. Then there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$.

Theorem 4.1.11 (Chebyshev's Inequality)**.** If $f \in L^p$ $(0 < p < \infty)$, then for any $\alpha > 0$, $\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$.

Definition 4.1.12. Let f, g be measurable functions on $\mathbb{R}^n$. Define the convolution of f and g, denoted $f * g(x)$, by $f * g(x) = \int f(x-y)g(y)\,dy$.

Theorem 4.1.13 (Young's Inequality)**.** If $f \in L^1$ and $g \in L^p$ $(1 \leq p \leq \infty)$, then $f * g(x)$ exists for almost every x, $f * g \in L^p$, and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

4.2 Hilbert Spaces

Definition 4.2.1. Let $\mathcal{H}$ be a complex vector space. An inner product on $\mathcal{H}$ is a map $(x, y) \to \langle x, y\rangle$ from $X \times X \to \mathbb{C}$ such that (i) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, x\rangle$ for all $x, y, z \in \mathcal{H}$, $a, b \in \mathbb{C}$. (ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in \mathcal{H}$. (iii) $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in \mathcal{H}$.

Definition 4.2.2. A Hilbert space is a complex vector space with inner product $\langle x, y\rangle$ which is complete with respect to the norm $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 4.2.3 (Schwarz Inequality)**.** Let $\mathcal{H}$ be a Hilbert space. Then $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{H}$, with equality iff x and y are linearly dependent.

Proposition 4.2.4 (Parallelogram Law)**.** For all $x, y \in \mathcal{H}$, the following holds: $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Definition 4.2.5. For $x, y \in \mathcal{H}$, say x is orthogonal to y, written $x \perp y$, if $\langle x, y\rangle = 0$. If $E \subset \mathcal{H}$, define $E^\perp = \{x \in \mathcal{H} : x \perp y, \forall\, y \in E\}$.

Theorem 4.2.6. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, i.e., each $x \in \mathcal{H}$ can be expressed uniquely as $x = y + z$, where $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$.

Theorem 4.2.7. If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle$.

Theorem 4.2.8 (Bessel's Inequality)**.** If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$, $\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2$.

Theorem 4.2.9. If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, the following are equivalent:

(a) (Completeness) If $\langle x, u_\alpha\rangle = 0$ for all α, then $x = 0$.

(b) (Parseval's Identity) $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ for all $x \in \mathcal{H}$.

(c) For each $x \in \mathcal{H}$, $x = \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha$, where the sum on the right has only countably many nonzero terms and converges in the norm topology no matter how the terms are ordered.

Remark 4.2.10. Have that L^2 is a Hilbert space with inner product $\langle f, g\rangle = \int f\overline{g}$.

4.3 Fourier Series

Definition 4.3.1. Define $\widehat{f}(n) = \langle \frac{1}{\sqrt{2\pi}}e^{inx}, f\rangle = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}e^{-inx}f(x)\,dx$.

Lemma 4.3.2 (Riemann-Lebesgue Lemma)**.** Let $f \in L^1(\mathbb{R})$. Then $\lim_{n\to\pm\infty} \widehat{f}(n) = 0$.

Theorem 4.3.3. Let $f \in C^2([0, 2\pi])$ be 2π periodic. Then $\sum_{j=-n}^{n} \widehat{f}(j)e^{ijx}$ converges to f uniformly in x as $n \to \infty$.

Corollary 4.3.4. Let $f \in L^2([0, 2\pi])$ be 2π periodic. Then $\sum_{j=-n}^{n} \widehat{f}(j)e^{ijx}$ converges to f in the L^2 norm.

Definition 4.3.5. Define the Dirichlet kernel, $D_n(x)$, to be equal to the sum $D_n(x) = \frac{1}{2\pi}\sum_{j=-n}^{n} e^{ijx}$.

Remark 4.3.6. Have $D_n(x) = \dfrac{1}{2\pi}\dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$.

Definition 4.3.7. Define the Poisson kernel, $P_\rho(x)$, to be equal to the sum $P_\rho(x) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} \rho^{|j|}e^{ijx}$.

Remark 4.3.8. (i) $P_\rho(x) \geq 0$. (ii) $P_\rho(x)$ is periodic in x and satisfies $\int_0^{2\pi} P_\rho(x)\,dx = 2\pi$. (iii) $P_\rho(x) = \dfrac{1}{\sqrt{2\pi}}\dfrac{1-\rho^2}{1-2\rho\cos x+\rho^2}$.

Theorem 4.3.9. Let f be continuous, 2π-periodic. Then $\displaystyle\lim_{\rho\nearrow 1}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\rho^{|j|}\widehat{f}(j)e^{ijx} = f(x)$.

Definition 4.3.10. Define the Fejer kernel, $F_n(x)$, to be the sum $\frac{1}{n+1}\sum_{m=0}^{n}\sum_{j=-m}^{m}\frac{1}{2\pi}e^{ijx}$.

Chapter 5

Additional Integration Topics

5.1 Modes of Convergence

Definition 5.1.1. Let $\{f_n\}$ be a sequence of complex-valued measurable functions on (X, μ). Say that $\{f_n\}$ is Cauchy in measure if $\forall\, \epsilon > 0$, $\mu\left(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}\right) \to 0$ as $m, n \to \infty$. Say that $\{f_n\}$ converges to f in measure if $\mu\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) \to 0$ as $n \to 0$.

Proposition 5.1.2. If $f_n \to f$ in L^1, then $f_n \to f$ in measure.

Proof. Let $\epsilon > 0$, and let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geq \epsilon\}$. Then $\int |f - f_n| \geq \int_{E_{n,\epsilon}} |f - f_n| \geq \epsilon\mu(E_{n,\epsilon})$. So $\mu(E_{n,\epsilon}) \leq \epsilon^{-1} \int |f - f_n| \to 0$ as $n \to \infty$. $\square$

Proposition 5.1.3. If $f_n \to f$ in measure, then some subsequence converges to f pointwise almost everywhere. In particular, if $f_n \to f$ in L^1, then some subsequence converges pointwise almost everywhere.

Theorem 5.1.4 (Egoroff's Theorem)**.** Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose f_n, f are measurable functions and $f_n \to f$ almost everywhere. Then for all $\epsilon > 0$, there exists $E \subseteq X$, $\mu(E) < \epsilon$, such that $f_n \to f$ uniformly on E^c.

Theorem 5.1.5 (Lusin's Theorem)**.** If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, then there is a compact set $E \subseteq [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

5.2 Signed Measures

Definition 5.2.1. Let $(X, \mathcal{M})$ be a measurable space. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that: (i) $\nu(\emptyset) = 0$, (ii) ν assumes at most one of the values $\pm\infty$, (iii) If $\{E_i\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\cup_1^\infty E_i) = \sum_1^\infty \nu(E_i)$, where the latter sum converges absolutely if $\nu(\cup_1^\infty E_i)$ is finite.

Definition 5.2.2. If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called positive (resp. negative, null) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0, \nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subset E$.

Theorem 5.2.3 (Hahn Decomposition Theorem)**.** If ν is a signed measure on $(X, \mathcal{M})$, then there exists a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \emptyset$. If P', N' is another such pair, then $P \Delta P'$ $(= N \Delta N')$ is null for ν.

Theorem 5.2.4 (Jordan Decomposition Theorem)**.** If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Definition 5.2.5. Let ν be a signed measure and let μ be a positive measure on $(X, \mathcal{M})$. Say that ν is absolutely continuous with respect to μ, denoted $\nu \ll \mu$, if $\nu(E) = 0$ for all $E \in \mathcal{M}$ such that $\mu(E) = 0$.

Theorem 5.2.6 (Lebesgue-Radon-Nikodym Theorem)**.** Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that $\lambda \perp \mu, \rho \ll \mu$, and $\nu = \lambda + \rho$. Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f\, d\mu$, and any two such functions are equal μ-a.e.

Definition 5.2.7. The function f above is called the Radon-Nikodym derivative of ν with respect to μ, and is denoted by $f = \frac{d\nu}{d\mu}$.

5.3 Absolutely Continuous Functions

Definition 5.3.1. A function $F : \mathbb{R} \to \mathbb{C}$ is called absolutely continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any finite set of disjoint open intervals $\{(a_i, b_i)\}_1^N$, if $\sum_1^N (b_i - a_i) < \delta$, then $\sum_1^N |F(b_i) - F(a_i)| < \epsilon$.

Proposition 5.3.2. Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded, increasing, right continuous function, and let μ_F be the associated Lebesgue-Stieltjes measure. Then F is absolutely continuous iff $\mu_F \ll m$.

Theorem 5.3.3 (Fundamental Theorem of Calculus)**.** If $-\infty < a < b < \infty$ and $F : [a, b] \to \mathbb{C}$, the following are equivalent:

(a) F is absolutely continuous on $[a, b]$.

(b) $F(x) - F(a) = \int_a^x f(t)\, dt$ for some $f \in L^1([a, b], m)$.

(c) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_a^x F'(t)\, dt$.

Part II

Complex Analysis

Chapter 6

Elementary Properties

6.1 Basic Concepts

Remark (Tartaglia's Method for Finding the Roots of Cubic Polynomials)**.** To solve the cubic polynomial $x^3 + ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$, set $y = x - a/3$. Simplfy the original equation to one of the form $y^3 + py = q$, where $p = b - a^2/3$ and $q = (9ab - 2a^3 - 27c)/27$. Look for a solution of the form $y = s^{1/3} - t^{1/3}$. We get $s + t + (p - 3s^{1/3}t^{1/3})(s^{1/3} - t^{1/3}) = q$. Requiring $s - t = q$, we have $p - 3s^{1/3}t^{1/3} = 0$, i.e., $st = (p/3)^3$. Then $s = t + q$ and $(t + q)t = (p/3)^3$, so $t = \frac{1}{2}\frac{-q+(q^2+4p^3/27)^{1/2}}{2}$, $s = \frac{1}{2}\frac{q+(q^2+4p^3/27)^{1/2}}{2}$, and

$$x = \left(\frac{q + (q^2 + \frac{4p^3}{27})^{\frac{1}{2}}}{2}\right)^{\frac{1}{3}} - \left(\frac{-q + (q^2 + \frac{4p^3}{27})^{\frac{1}{2}}}{2}\right)^{\frac{1}{3}} - \frac{1}{3a}$$

Remark 6.1.1 (Elementary concepts)**.** Definition of complex numbers. Addition and multiplication of complex numbers. Complex conjugates. Absolute value of complex numbers. Inverses of non-zero complex numbers.

Remark 6.1.2 (Failure of an Ordering)**.** The complex numbers do not form an ordered field. A field $\mathbb{F}$ is an ordered field if there exists a subset $P \subset \mathbb{F}$ such that (i) $\forall v \in \mathbb{F}$, $v \in P$ or $-v \in P$, and (ii) $\forall v \in \mathbb{F}$, $v^2 \in P$. Suppose such a $P \subset \mathbb{C}$ exists. Then either $1 \in P$ or $-1 \in P \Rightarrow (1)^2 = (-1)^2 = 1 \in P$. And also, either $i \in P$ or $-i \in P \Rightarrow (i)^2 = (-i)^2 = -1 \in P$, i.e., $1 \in P$ and $-1 \in P$, $(\Rightarrow\Leftarrow)$.

Theorem 6.1.3 (Inequalities for $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $|z|$)**.** Let $z, w \in \mathbb{C}$. Then the following properties hold:

1. $|\operatorname{Re}(z)| \leq |z|$

2. $|\mathrm{Im}\,(z)| \leq |z|$

3. $|z+w| \leq |z| + |w|$

4. $||z| - |w|| \leq |z-w|$

Proof. Let $z = a + ib$, $a, b \in \mathbb{R}$.

1. Since $b^2 \geq 0$, $|\,\mathrm{Re}\,(z)\,| = |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2}$.

2. Since $a^2 \geq 0$, $|\,\mathrm{Im}\,(z)\,| = |b| = \sqrt{b^2} \leq \sqrt{a^2+b^2}$.

3. Have that $|z+w|^2 = (z+w)\overline{(z+w)} = |z|^2 + |w|^2 + 2\,\mathrm{Re}\,(zw) \leq |z|^2 + |w|^2 + 2|zw| = (|z|+|w|)^2$, so $|z+w| \leq |z| + |w|$.

4. Have that $|z| = |z - w + w| \leq |z-w| + |w|$, so $|z| - |w| \leq |z-w|$. Similarly, $|w| = |w - z + z| \leq |w-z| + |z| = |z-w| + |z|$, so $|w| - |z| \leq |z-w|$. Then $||z| - |w|| \leq |z-w|$. ☐

Remark 6.1.4 (Polar Representation of Complex Numbers)**.** Given a complex number $z = x + iy$, we can express z in polor notation as $z = |z|\, e^{i \arg z}$, where as usual $|z| = x^2 + y^2$, and $\arg z = \arctan y/x$. Note that for a real number θ, the expression $e^{i\theta}$ is given by Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$.

Remark 6.1.5. Geometric interpretation of complex multiplication.

Remark 6.1.6 (Roots of Complex Numbers)**.** Given a complex number $z = re^{i\theta}$, we compute the n-th roots of z as $\sqrt[n]{z} = \sqrt[n]{r}e^{i(\theta/n + 2\pi k/n)}$ for $k = 0, 1, 2, \ldots, (n-1)$.

6.2 Topological Properties

Remark 6.2.1 (Topological Properties of $\mathbb{C}$)**.** The function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by $d(z, w) = |z - w|$ defines a metric on $\mathbb{C}$. We give $\mathbb{C}$ the metric topology induced by the metric $d(z, w) = |z - w|$.

Definition 6.2.2. A set $X \subset \mathbb{C}$ is connected if whenever $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, where $X_1 = X \cap G$, and $X_2 = X \cap G$, G_1 and G_2 open, either $X_1 = \emptyset$ or $X_2 = \emptyset$. Call X arc-connected if every two points in X can be joined by a polygonal path in X.

Definition 6.2.3. A region in $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$.

Theorem 6.2.4. If $X \subset \mathbb{C}$ is open, then X is connected iff X is arc-connected.

Proof. ($\Rightarrow$) Assume X is open and connected. Let $z_0 \in X$. Let $X_1 = \{z \in X : z$ and z_0 can be joined by a polygonal path in $X\}$. Then X_1 is open. Let $X_2 = X \backslash X_1$. Then X_2 is open as well. Since $X_1 \neq \emptyset$ ($z_0 \in X_1$), we have $X_2 = \emptyset$ and $X_1 = X$.

($\Leftarrow$) Assume X is arc-connected. Suppose X can be decomposed as the union of two non-empty open sets G_1 and G_2 with $G_1 \cap G_2 = \emptyset$. Let $\gamma(t) : [0,1] \to \mathbb{C}$ be a polygonal path with $\gamma(0) \in G_1$ and $\gamma(1) \in G_2$. Then $[0,1]$ decomposes as the disjoint union of the open sets $\gamma^{-1}(G_1)$ and $\gamma^{-1}(G_2)$, ($\Rightarrow\Leftarrow$). Conclude that X must be connected. □

Definition 6.2.5 (Extended Complex Plane $\mathbb{C}_\infty$ and Stereographic Projection)**.** We have the natural embedding of S^2 into $\mathbb{R}^3 = \{(x_1, x_2, x_3) | \, x_1, x_2, x_3 \in \mathbb{R}\}$. Embed $\mathbb{C}$ in the x_1, x_2-plane. Identify the complex number $z = x + iy$ with the point (x_1, x_2, x_3) on S^2 that is the intersection of S^2 with the line passing through the points $Z = (x, y, 0)$ and $N = (0, 0, 1)$. Given a point of intersection (x_1, x_2, x_3), we have $z = \frac{x_1 + ix_2}{1 - x_3}$. Given the complex number $z = x + iy$, we have

$$x_1 = \frac{z + \bar{z}}{|z|^2 + 1} \qquad x_2 = \frac{-i(z - \bar{z})}{|z|^2 + 1} \qquad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

We identify the point $\infty \in \mathbb{C}_\infty$ with the point $N = (0, 0, 1)$.

Definition 6.2.6 (Topology on $\mathbb{C}_\infty$)**.** The topology on $\mathbb{C}_\infty$ is the metric topology on S^2 induced from $\mathbb{R}^3$. We say $z_n \to \infty$ if $|z_n| \to \infty$, and we say $\lim_{z \to \infty} f(z) = L$ if $\exists \, R > 0$ such that $\forall \, \epsilon > 0$, $|f(z) - L| < \epsilon$ when $|z| > R$.

Chapter 7

Analytic and Harmonic Functions

7.1 Basic Properties of Analytic Functions

Definition 7.1.1. Let $f : G \to \mathbb{C}$ where $G \subset \mathbb{C}$ is open. Say f is differentiable at $a \in G$ if $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$ exists and is finite. Call f analytic (holomorphic) if it is continuously differentiable at every point $a \in G$.

Theorem 7.1.2. Let G be a region, and let $f, g : G \to \mathbb{C}$ be analytic. Let $a, b \in \mathbb{C}$.

1. $af + bg$ is analytic on G and $(af + bg)' = af' + bg'$
2. fg is analytic on G and $(fg)' = f'g + fg'$
3. Suppose g is non-zero on G. Then f/g is analytic on G and $(f/g)' = \dfrac{f'g - fg'}{g^2}$

Proof.

1. $\lim_{z\to z_0} \frac{(af+bg)(z)-(af+bg)(z_0)}{z-z_0} = \lim_{z\to z_0} a\frac{f(z)-f(z_0)}{z-z_0} + \lim_{z\to z_0} b\frac{g(z)-g(z_0)}{z-z_0} = af'(z_0) + bg'(z_0)$
2. $\lim_{z\to z_0} \frac{(fg)(z)-(fg)(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}g(z) + \lim_{z\to z_0} f(z)\frac{g(z)-g(z_0)}{z-z_0} = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
3. $\lim_{z\to z_0} \left(\frac{f}{g}(z) - \frac{f}{g}(z_0)\right)(z-z_0)^{-1} = \lim_{z\to z_0} \left(\frac{f(z)g(z_0)-g(z)f(z_0)}{g(z)g(z_0)}\right)(z-z_0)^{-1}$
   $= \frac{g(z_0)}{g(z_0)} \lim_{z\to z_0} \frac{1}{g(z)} \frac{f(z)-f(z_0)}{z-z_0} - \frac{f(z_0)}{g(z_0)} \lim_{z\to z_0} \frac{1}{g(z)} \frac{g(z)-g(z_0)}{z-z_0} = \frac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{g(z_0)^2}$ □

Example 7.1.3. Polynomials with complex coefficients are analytic on all of $\mathbb{C}$. Rational functions $\frac{f}{g}$ are analytic where $g \neq 0$.

Theorem 7.1.4 (Chain Rule)**.** Let G be a region, let f be analytic on G, and let g be analytic on $f(G)$. Then $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))f'(z)$.

Proof. Let $z_0 \in G$, and let $w_0 = f(z_0)$. For $w \in f(G)$, set

$$h(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} - g'(w_0) & w \neq w_0 \\ 0 & w = w_0 \end{cases}$$

Then h is continuous. Since the composition of continuous functions is continuous, $h \circ f$ is continuous and $\lim_{z \to z_0} (h \circ f)(z) = 0$. Now

$$(g \circ f)(z) - g(w_0) = [(h \circ f)(z) + g'(w_0)]\,(f(z) - w_0)$$

Then for $z \neq z_0$,

$$\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = [(h \circ f)(z) + g'(w_0)]\,\frac{f(z) - f(z_0)}{z - z_0}$$

and taking the limit as $z \longrightarrow z_0$, the right side of the equation converges to $[0 + g'(w_0)]\,f'(z_0)$, yielding the desired relation. ☐

Example 7.1.5. The function $f(z) = \bar{z}$ is not analytic because it does not satisfy the Cauchy-Riemann Equations (see below).

Remark 7.1.6 (Cauchy-Riemann Equations)**.** Let $f : G \to \mathbb{C}$, and write $f = u + iv$ where $u, v : G \to \mathbb{R}$ satisfty $u = \mathrm{Re}\,(f)$ and $v = \mathrm{Im}\,(f)$. Suppose f is differentiable at $z = x + iy \in G$. Then

$$f'(z) = \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x+h, y) - v(x, y)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

and

$$f'(z) = \lim_{ih \to 0} \frac{u(x, y+h) - u(x, y)}{ih} + i \lim_{ih \to 0} \frac{v(x, y+h) - v(x, y)}{ih} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts from these two equations, we obtain the Cauchy Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \quad \text{or, equivalently} \quad \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

Theorem 7.1.7. Let G be an open set, $f : G \to \mathbb{C}$, $f = u + iv$.

1. If f is analytic on G, then u and v have first partial derivatives satisftying the Cauchy-Riemann Equations.

2. If u and v have continuous first partial derivatives satisfying the Cauchy-Riemann Equations, then f is analytic on G.

Proof.

1. Done by the previous Remark.
2. Fix $z \in G$. Show $\lim_{h+ik \to 0} \frac{f(z+h+ik)-f(z)}{h+ik}$ exists. *Claim*:

$$\begin{aligned} u(x+h, y+k) - u(x,y) &= \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + (|h| + |k|)\epsilon_1 \\ v(x+h, y+k) - v(x,y) &= \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + (|h| + |k|)\epsilon_2 \end{aligned}$$

where $\epsilon_j = \epsilon_j(h,k) \longrightarrow 0$ as $|h| + |k| \longrightarrow 0$, $j = 1, 2$.
Reasoning: Set $\varphi(t) = u(x + th, y + tk)$, $0 \le t \le 1$. Since the partial derivatives of u are continuous, u is continuously differentiable by the results of Multivariable Calculus. (In particular, u is continuous.) By the Mean Value Theorem, $\exists \, \tau \in (0,1)$ such that $\varphi(1) - \varphi(0) = \varphi'(\tau)(1-0) = \varphi'(\tau)$. Then

$$\begin{aligned} u(x+h, y+k) - u(x,y) &= u_x(x + \tau h), y + \tau k)h + u_y(x + \tau h, y + \tau k)k \\ &= \frac{\partial u}{\partial x}(x,y)h + \frac{\partial u}{\partial y}(x,y)k + (|h| + |k|)\epsilon_1 \end{aligned}$$

where

$$\epsilon_1 = \frac{1}{|h| + |k|} \left[\left(u_x(x+h, y+\tau k) - u_x(x,y) \right) h + \left(u_y(x+h, y+k) - u_y(x,y) \right) k \right]$$

Here $\epsilon_1 \longrightarrow 0$ as $|h| = |k| \longrightarrow 0$, and the claim follows by a similar argument for $v(x+h, y+k) - v(x,y)$. Now, applying the claim and since u and v satisfy the Cauchy-Riemann Equations,

$$\frac{f(z+h+ik) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + \frac{|h| + |k|}{h+ik}(\epsilon_1 + i\epsilon_2)$$

Since $\left| \frac{|h| + |k|}{h+ik} \right| \le 2$, we have that $\frac{|h| + |k|}{h+ik}(\epsilon_1 + i\epsilon_2) \longrightarrow 0$ as $|h| + |k| \longrightarrow 0$, and the result follows. ☐

Lemma 7.1.8. Let G be a region and $u : G \to \mathbb{R}$. If $u_x = u_y = 0$ on G, then u is constant.

Proof. Suppose G is an open disk. Let $(x_1, y_1), (x_2, y_2) \in G$. Set $\varphi(t) = u(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$, $0 \le t \le 1$. By the Mean Value Theorem, $\exists \, \tau \in (0,1)$ such that

$\varphi(1) - \varphi(0) = \varphi'(\tau)$. Then

$$\begin{aligned} u(x_2, y_2) - u(x_1, y_1) &= u_x(x_1 + \tau(x_2 - x_1), y_1 + \tau(y_2 - y_1))(x_2 - x_1) \\ &\quad + u_y(x_1 + \tau(x_2 - x_1), y_1 + \tau(y_2 - y_1))(y_2 - y_1) \\ &= 0(x_2 - x_1) + 0(y_2 - y_1) = 0 \end{aligned}$$

Conclude that u is constant. ☐

Theorem 7.1.9. If f is analytic on a region G and $f'(z) = 0$ for all $z \in G$, then f is constant.

Proof. Have that $0 \equiv f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$. Then $u_x = u_y = v_x = v_y \equiv 0$, so by the previous lemma each of u and v are constant, so f is contstant. ☐

Theorem 7.1.10. *Variant 1*: Let f be analytic on a region G. If f is real-valued, then f is constant. *Variant 2*: Let f, g be analytic on a region G, $\mathrm{Re}\,(f) = \mathrm{Re}\,(g)$ on G. Then $f = g + ic$ for some constant $c \in \mathbb{R}$. *Variant 3*: Let f be analytic on a region G. Suppose f takes its values on a line $ax + by + c = 0$ with $|a| + |b| > 0$. Then f is constant.

Proof of Variant 1. Write $f = u + iv$, $v \equiv 0$. Then $v_x = v_y = 0$, so by the Cauchy-Riemann Equations, $u_x = u_y = 0$. The result then follows by the previous theorem. ☐

Proof of Variant 2. The result follows from the previous theorem after considering the analytic, real-valued function $\frac{1}{i}(f - g)$. ☐

Proof of Variant 3. Write $f = u + iv$. Then $au + bv + c = 0$, and $au_x + bv_x = 0$, $au_y + bv_y = 0$. Applying the Cauchy-Riemann Equations we can rewrite this system of equations as $au_x + bv_x = 0$, $bu_x - av_x = 0$. Since $a^2 + b^2 \neq 0$, we must have $u_x = v_x = 0$, which implies by the Cauchy-Riemann Equations that $u_y = v_y = 0$, and hence f is constant by the previous result. ☐

7.2 Basic Properties of Harmonic Functions

Definition 7.2.1. A function $u : G \to \mathbb{R}$ is called harmonic if it satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

Example 7.2.2. Let $f = u + iv$ be analytic on the region G. Assume that u and v have continuous first partial derivatives up to the second order. Then u and v are harmonic. (Note: The assumption that u and v have continuous first partial derivatives up to the second order later becomes redundant.)

Proof. We have that

$$\begin{aligned}
u_{xx}+u_{yy} &= (u_x)_x+(u_y)_y \\
&= (v_y)_x+(-v_x)_y=0 \quad \text{by Multivariable Calculus} \\
v_{xx}+v_{yy} &= (v_x)_x+(v_y)_y \\
&= (-u_y)_x+(u_x)_y=0 \quad \text{by Multivariable Calculus}
\end{aligned}$$

□

Definition 7.2.3. If $u:G\to\mathbb{R}$ is harmonic on a region G and $f=u+iv$ is analytic, call v the harmonic conjugate to u.

Remark. Harmonic conjugates to $u:G\to\mathbb{R}$ are unique only up to additive constants.

Remark 7.2.4. The existence of a harmonic conjugate to $u:G\to\mathbb{R}$ depends on the region G. For instance, $u(x,y)=\frac{1}{2}\log(x^2+y^2)$ has no harmonic conjugate on $G=\mathbb{C}\backslash\{0\}$.

Lemma 7.2.5 (Leibnitz's Rule)**.** *Let $\varphi(x,y)$ and $\varphi_x(x,y)$ be continuous on $[a,b]\times[c,d]$. Then $f(x)=\int_c^d\varphi(x,y)dy$ is continuous for $a\le x\le b$, differentiable for $a<x<b$, and $f'(x)=\int_c^d\varphi_x(x,y)dy$.*

Proof. Since $\varphi(x,y)$ and $\varphi_x(x,y)$ are both continuous on $[a,b]\times[c,d]$, they are each measurable. Because continuous functions on compact sets are bounded, $\exists\, M>0$ such that $|\varphi(x,y)|\le M$ and $|\varphi_x(x,y)|\le M$, $\forall\,(x,y)\in[a,b]\times[c,d]$. Then by the Dominated Convergence Theorem with the dominating function $g:[a,b]\times[c,d]\to\mathbb{R}$ defined by $g(x,y)\equiv M$, we have that $f(x)=\int_c^d\varphi(x,y)dy$ is continuous for $a\le x\le b$, is differentiable for $a<x<b$, and $f'(x)=\int_c^d\varphi_x(x,y)dy$. □

Theorem 7.2.6. *If G is an open disc, a rectangle, or $\mathbb{C}$, then every harmonic function $u:G\to\mathbb{R}$ has a harmonic conjugate.*

Proof. Assume $G=D(0,r)$. (Otherwise perform an appropriate translation.) Let $v(x,y)=\int_0^y u_x(x,t)dt-\int_0^x u_y(s,0)ds$. Then $v_y=u_x(x,y)$. By the previous theorem (Leibnitz's Rule), we have

$$\begin{aligned}
v_x(x,y) &= \int_0^y u_{xx}(x,t)dt-u_y(x,0)=-\int_0^y u_{yy}(x,t)dt-u_y(x,0) \\
&= -\int_0^y \frac{d}{dt}u_y(x,t)dt-u_y(x,0)=-u_y(x,y)
\end{aligned}$$

Thus the Cauchy-Riemann equations are satisfied, and $f=u+iv$ is analytic by Theorem 2.7 (2). □

Chapter 8

Power Series and Analytic Functions

8.1 Properties of Power Series

Theorem (Weierstrass M-Test)**.** For $k \in \mathbb{N}$, let $a_k : G \to \mathbb{C}$. Suppose that for each $k \in \mathbb{N}$, $\exists\, M_k > 0$ such that $|a_k(z)| \leq M_k$, $\forall\, z \in G$, and $\sum_{k=1}^{\infty} M_k < \infty$. Then $\sum_{k=1}^{\infty} a_k(z)$ converges absolutely and uniformly on G.

Proof. Let $\epsilon > 0$. To prove absolute and uniform convergence of $\sum_{k=1}^{\infty} a_k(z)$, it suffices to show that $\exists\, N \in \mathbb{N}$ such that $\forall\, n, m \geq N$, $n \leq m$, $|\sum_{k=n}^{m} |a_k(z)|| < \epsilon$ for all $z \in G$, for then the sequence of functions $S_N(z) := \sum_{k=1}^{N} |a_k(z)|$ is uniformly Cauchy on G, and hence converges uniformly as $N \to \infty$. Now, since $\sum_{k=1}^{\infty} M_k < \infty$, let $N \in \mathbb{N}$ such that for all $n \geq N$, $\sum_{k=n}^{\infty} M_k < \epsilon$. Then for all $n, m \geq N$, $n \leq m$, $|\sum_{k=n}^{m} |a_k(z)|| \leq \sum_{k=n}^{m} |a_k(z)| \leq \sum_{k=n}^{m} M_k < \epsilon$, for all $z \in G$. □

Theorem 8.1.1 (Cauchy Product of Series)**.** Suppose $A = \sum a_k, B = \sum b_k$ are absolutely convergent series. Let $c_k = \sum_{i=1}^{k} a_i b_{k-i}$. Then $C = \sum_{k=0}^{\infty} c_k$ is absolutely convergent and $C = AB$.

Proof. Let $d_k = \sum_{i=0}^{k} |a_i|\,|b_{k-i}|$, $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, and $C_n = \sum_{k=0}^{n} c_k$. Then

$$\sum_{k=0}^{n} d_k \leq (\sum_{i=0}^{n} |a_i|)(\sum_{i=0}^{\infty} |b_i|) \leq (\sum_{i=0}^{\infty} |a_i|)(\sum_{i=0}^{\infty} |b_i|) < \infty$$

So $\sum_{k=0}^{\infty} d_k < \infty$. Now

$$A_n B_n = C_n + \sum_{k=n+1}^{2n} (\sum_{p+q=k} a_p b_q) = C_n + R_n$$

where $R_n = \sum_{k=n+1}^{2n}(\sum_{p+q=k} a_p b_q)$. But

$$|R_n| \leq \sum_{k=n+1}^{2n} (\sum_{p+q=k} |a_p||b_q|) = \sum_{k=n+1}^{2n} d_k$$

Since $\sum_{k=0}^{\infty} d_k < \infty$, we must have $\lim_{n\to\infty} \sum_{k=n+1}^{2n} d_k = 0$. We conclude $\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} C_n$. □

Definition 8.1.2. A power series is a series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$ where $a, c_n \in \mathbb{C}$.

Theorem 8.1.3 (Abel's Theorem)**.** Given a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ where $a, c_n \in \mathbb{C}$, define $R := \sup\{r \in \mathbb{R} : |c_n|\, r^n \text{ is bounded for all } n \in \mathbb{N}\}$. Then one of the following holds:

1. $R = 0$ and the series converges for $z = a$ and diverges for $z \neq a$.
2. $0 < R < \infty$ and the series converges uniformly and absolutely on all compact subsets of $D(a, R)$, and diverges for $|z - a| > R$.
3. $R = \infty$ and the series converges absolutely and uniformly on all compact subsets of $\mathbb{C}$.

Proof. Take $a = 0$. (Otherwise perform an appropriate translation.)

- Suppose $R = 0$. The the series converges (to c_0) for $z = a$.
- Suppose $0 < R \leq \infty$. Let $r_0 \in (0, R)$. Choose $r_1 \in (r_0, R)$. By the definition of R, $\exists M > 0$ such that for all $n \in \mathbb{N}$, $|c_n| r_1^n \leq M$. Let $z \in \{z : |z| \leq r_0\}$. Then for all $n \in \mathbb{N}$, $|c_n z^n| = |c_n||z|^n \leq |c_n| r_0^n = |c_n| r_1^n \left(\frac{r_0}{r_1}\right)^n \leq M \left(\frac{r_0}{r_1}\right)^n$. Since $\frac{r_0}{r_1} < 1$, $\sum_{n=0}^{\infty} M \left(\frac{r_0}{r_1}\right)^n < \infty$. Then $\sum_{n=0}^{\infty} c_n z^n$ converges by the Weierstrass M-test, and $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely and uniformly on all compact subsets of $D(0, R)$.
- Note that if $\sum_{n=0}^{\infty} c_n z^n < \infty$, then $\lim_{n\to\infty} |c_n z^n| = 0$. Suppose $R < \infty$ and $|z| > R$. Then by the definition of R the terms $|c_n z^n|$ are unbounded as $n \to \infty$, and hence $\sum_{n=0}^{\infty} c_n z^n$ diverges. □

Remark. Above, R is called the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$.

Remark 8.1.4. Recall that given a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ the limit superior and limit inferior of the sequence are defined by $\limsup a_n := \lim_{k\to\infty} \sup_{n\geq k} a_n$ and $\liminf a_n := \lim_{k\to\infty} \inf_{n\geq k} a_n$, respectively.

Lemma. Consider the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ with radius of convergence R.

1. The radius of convergence R is given by the formula $\frac{1}{R} = \limsup |c_n|^{1/n}$.

2. The limit $\lim_{n\to\infty} \dfrac{|c_n|}{|c_{n+1}|}$, if it exists, is equal to R.

3. If $\rho = \lim_{n\to\infty} \sqrt[n]{|c_n|}$ exists, then $\rho = \frac{1}{R}$.

Proof.

1. If $R = 0$, then $\{|c_n|\}_{n=0}^{\infty}$ is an unbounded sequence, so given any $M > 0$, $|c_n| > M$ for infinitely many $n \in \mathbb{N}$. Then $\forall\, M > 0$, $|c_n|^{1/n} > M^{1/n}$ for infinitely many $n \in \mathbb{N}$, so $\limsup |c_n|^{1/n} \geq M^{1/n}$ for all $M > 0$, i.e., $\limsup |c_n|^{1/n} = \infty = \frac{1}{R}$. Now suppose $0 < R \leq \infty$, and let $r \in (0, R)$. By the definition of R, $\exists\, M > 0$ such that $|c_n| r^n \leq M$ for all $n \in \mathbb{N}$. Then $|c_n|^{1/n} \leq \frac{M^{1/n}}{r}$. Since $M^{1/n} \to 1$ as $n \to \infty$, $\limsup |c_n|^{1/n} \leq \frac{1}{r}$ for all $r \in (0, R)$. If $R = \infty$, we have that $\limsup |c_n|^{1/n} \leq \frac{1}{r}$ for all $r > 0$, i.e., $\limsup |c_n|^{1/n} = 0 = \frac{1}{R}$. Suppose $R < \infty$, and let $r \in (R, \infty)$. Then $|c_n| r^n \geq 1$ for infinitely many $n \in \mathbb{N}$, so $|c_n|^{1/n} \geq \frac{1}{r}$ for infinitely many $n \in \mathbb{N}$, from which we conclude that $\limsup |c_n|^{1/n} \geq \frac{1}{r}$ for all $r \in (R, \infty)$. Now by the previous remarks, $\limsup |c_n|^{1/n} \leq \frac{1}{R} \leq \limsup |c_n|^{1/n}$, i.e., $\limsup |c_n|^{1/n} = \frac{1}{R}$.

2. Suppose that $\lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}$ exists. By the Ratio Test for series of real numbers, we know that $\sum_{n=0}^{\infty} |c_n| r^n$ converges if $\lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} > r$, and diverges if $\lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} < r$. Then $\lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = R$ by the definition of R.

3. Suppose $\rho = \lim_{n\to\infty} \sqrt[n]{|c_n|}$ exists. By the Root Test for series of real numbers, we know that $\sum_{n=0}^{\infty} |c_n| r^n$ converges if $r < 1/\left(\lim_{n\to\infty} |c_n|^{1/n}\right)$, and diverges if $r > 1/\left(\lim_{n\to\infty} |c_n|^{1/n}\right)$. Then $\rho = \frac{1}{R}$ by the definition of R. ☐

Example 8.1.5. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{3^n (z-1)^{2n}}{n+1}$.

Remark. $n^{1/n} \to 1$ as $n \to \infty$.

Proof. Write $n^{1/n} = 1 + \delta_n$. Then $n = (1+\delta_n)^n = 1 + \binom{n}{1}\delta_n + \binom{n}{2}\delta_n^2 + \ldots + \binom{n}{n}\delta_n^n \geq 1 + \binom{n}{2}\delta_n^2$. Then $\delta_n^2 \leq \frac{n-1}{\binom{n}{2}} = \frac{2}{n} \to 0$ as $n \to \infty$. ☐

Theorem 8.1.6 (Sums and Products of Power Series)**.** Given two power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ with radii of convergence R, the power series $\sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n$ and $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ where $c_n = \sum_{i=1}^{n} a_i b_{n-i}$ both converge and have radii of convergence $\geq R$.

Theorem 8.1.7 (Differentiation of Power Series)**.** Suppose $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ has radius of convergence R. Then f is analytic on $D(a, R)$, $\sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$ has radius of convergence R, and $f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$ on $D(a, R)$. Also, $c_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Take $a = 0$. (Otherwise apply an appropriate translation.) Let R' be the radius of convergence of $\sum_{n=1}^{\infty} nc_n z^{n-1}$. This series has the same radius of convergence as $z\left(\sum_{n=1}^{\infty} nc_n z^{n-1}\right) = \sum_{n=1}^{\infty} nc_n z^n$. We have

$$\limsup |nc_n|^{1/n} = \limsup n^{1/n}|c_n|^{1/n} = \limsup |c_n|^{1/n}$$

by the remark above. Conclude that $R' = R$. Now show $\lim_{z\to w} \frac{f(z)-f(w)}{z-w} = g(w)$, where $g(z) = \sum_{n=1}^{\infty} nc_n z^{n-1}$. Let $z, w \in \mathbb{C}$, $|z| < p$, $|w| < p$, $0 < p < R$. Then

$$\begin{aligned} \left|\frac{f(z)-f(w)}{z-w} - g(w)\right| &= \left|\sum_{n=0}^{\infty} c_n \left(\frac{z^n - w^n}{z-w} - nw^{n-1}\right)\right| \\ &\leq \sum_{n=2}^{\infty} |c_n| \left|\frac{z^n - w^n}{z-w} - nw^{n-1}\right| \end{aligned}$$

Note that $\frac{z^n-w^n}{z-w} - nw^{n-1} = (z-w)(z^{n-2} + 2wz^{n-3} + 3w^2z^{n-4} + \cdots + (n-1)w^{n-2})$. Then

$$\begin{aligned} \left|\frac{z^n - w^n}{z-w} - nw^{n-1}\right| &\leq |z-w|(p^{n-2} + 2p^{n-2} + 3p^{n-2} + \cdots + (n-1)p^{n-2}) \\ &= |z-w|\frac{n(n-1)}{2}p^{n-2} \\ &\leq n^2p^{n-2}|z-w| \end{aligned}$$

and we have

$$\begin{aligned} \left|\frac{f(z)-f(w)}{z-w} - g(w)\right| &\leq \left(\sum_{n=2}^{\infty} |c_n| n^2 p^{n-2}\right)|z-w| \\ &= Mp|z-w| \end{aligned}$$

with $M < \infty$ by the first part of the proof. Now taking the limit as $z \to w$, we conclude $g(z) = f'(z)$. $\square$

8.2 Analytic Functions

Definition 8.2.1 (Exponential and Trigonometric Functions)**.** Set $\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$, $\sin z = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}$, $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. These functions are all (everywhere) analytic by the Ratio Test. Furthermore, we have the properties $\frac{d}{dz}e^z = e^z$, $\frac{d}{dz}\cos z = -\sin z$, $\frac{d}{dz}\sin z = \cos z$, and the identities $e^{iz} = \cos z + i\sin z$, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.

Definition 8.2.2 (The Complex Logarithm Function)**.**

1. By $\log z$ we mean the multivalued function $\log z := \log|z| + i\arg z$, where $\arg z$ denotes any value of the argument of z.

2. Let G be a region, $0 \notin G$, and let $f : G \to \mathbb{C}$. Call f a branch of the logarithm if f is continuous and $\exp f(z) = z$, $\forall z \in G$. Taking $G = \mathbb{C}\backslash(-\infty, 0]$, we call $f(z) = \log z + i\arg z$ with $|\arg z| < \pi$ the principal branch of the logarithm.

Theorem 8.2.3. Let $f(z)$ be a branch of $\log z$ on a region G with $0 \notin G$. Then f is analytic on G and $\frac{d}{dz}\log z = \frac{1}{z}$.

Proof. Fix $a \in G$, $\delta > 0$ such that $D(a,\delta) \subset G$. Suppose $0 < |h| < \delta$. Then $a + h \in D(a,\delta)$. Then $\frac{e^{f(a+h)} - e^{f(a)}}{h} = \frac{a+h-a}{h} = 1$. Hence $f(a+h) \neq f(a)$. Now $1 = \frac{e^{f(a+h)} - e^{f(a)}}{f(a+h) - f(a)} \frac{f(a+h) - f(a)}{h}$. Taking the limit as $h \to 0$, we have $1 = \left(\frac{d}{dz} e^z \big|_{z=f(a)}\right) \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Then $1 = a\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)$, and we conclude $f'(a) = \frac{1}{a}$. □

Remark. The identity $\log z_1 z_2 = \log z_1 + \log z_2$ is true only up to integral multiples of $2\pi i$. Take for instance $z_1 = z_2 = \exp\frac{3\pi i}{4}$ and the principal branch of the logarithm. Then $\frac{3\pi i}{2} = \log z_1 + \log z_2 \neq \log z_1 z_2 = \log\exp\left(\frac{3\pi i}{2}\right) = \log\exp\left(-\frac{\pi i}{2}\right) = -\frac{\pi i}{2}$.

Definition 8.2.4 (General Powers)**.** Given $a, b \in \mathbb{C}$, $a \neq 0$, define $a^b := e^{b\log a}$. (This function is possibly multivalued.)

Example 8.2.5 (Other Multivalued Functions)**.** Some other multivalued functions include $z^{1/n}$, $\sqrt{-z}$, and $\sqrt[3]{1+z^2}$. Given a region G, by a branch of $z^{1/n}$ on G we mean an analytic function $f : G \to \mathbb{C}$ such that $f(z)^n = z$, $\forall\, z \in G$.

Chapter 9

Conformal Mappings

9.1 Basic Properties

Remark 9.1.1 (Analytic Functions as Mappings)**.** Fix $z_0 \in G$, f analytic on G with $f'(z_0) \neq 0$. Consider a differentiable curve $C : \gamma(t) = x(t) + iy(t)$, $a < t < b$, with $\gamma(t_0) = z_0$. Assume $\gamma'(t_0) \neq 0$, so a tangent vector exists. Let $\theta_c = \arg \gamma'(t_0)$. The image of C under f is $(f \circ \gamma)(t)$, which satisfies $(f \circ \gamma)(t_0) = f(z_0)$, and $(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$. Let $\theta_{c'} = \arg(f \circ \gamma)'(t_0)$.

- **Property 1:** A curve C meeting z_0 at an angle θ_c with the positive real direction is mapped to a curve C' meeting $f(z_0)$ at an angle $\theta_{c'} = \theta + \alpha$ where $\alpha = \alpha(z_0)$ is independent of C ($\alpha = \arg f'(z_0)$).

- **Property 2:** Line segments at z_0 are expanded linearly with a ratio $p = p(z_0)$, $0 < p < \infty$, in the sense that $\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = p$.

Definition 9.1.2. A mapping $f : G \to \mathbb{C}$ with Properties **1** and **2** above at $z_0 \in G$ is called (directly) conformal at z_0. It is conformal on G if it is conformal at every point in G. The map is called indirectly conformal if Properties **1** and **2** hold with $\theta_{c'} = \theta + \alpha$ replaced by $\theta_{c'} = \theta - \alpha$.

Theorem 9.1.3. The mapping $w = f(z)$ defined by an analytic function $f : G \to \mathbb{C}$ is conformal at any point z_0 such that $f'(z_0) \neq 0$. It is conformal on G if $f'(z) \neq 0$, $\forall\, z \in G$.

Remark. The converse of the above theorem is true.

Remark. If $f : G \to \mathbb{C}$ is analytic, then $w = \overline{f(z)}$ is indirectly conformal $\forall\, z \in G$ such that $f'(z) \neq 0$.

Lemma 9.1.4. Let $\gamma(t)$ be a differentiable curve with values in some region G. If $f : G \to \mathbb{C}$ is analytic, then $\varphi(t) = (f \circ \gamma)(t)$ is differentiable and $\varphi'(t) = f'(\gamma(t))\gamma'(t)$.

Proof. Same as the proof of 2.4. $\square$

Remark 9.1.5 (Analytic Functions are Locally Linear)**.** Let $f : G \to \mathbb{C}$ be analytic, $z_0 \in G$, $f'(z_0) \neq 0$. Then $\frac{f(z)-f(z_0)}{z-z_0} = f'(z) + \epsilon(z, z_0)$ where $\epsilon(z, z_0) \to 0$ as $z \to z_0$. Near z_0, $f(z) \approx L(z) = f(z_0) + f'(z_0)(z - z_0)$, the composition of three linear operators: $z \overset{L_1}{\mapsto} z - z_0 \overset{L_2}{\mapsto} f'(z_0)(z - z_0) \overset{L_3}{\mapsto} L(z)$. Key steps at L_2: rotate and expand. In particular:

1. The angle between curves is preserved by conformal maps.
2. A small figure at z_0 will be rotated and slightly rotated, but its features preserved.

9.2 Properties of Linear Fractional Transformations

Definition 9.2.1 (Linear Fractional Transformations)**.** Let $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. Set $w = Tz = \frac{az+b}{cz+d}$. Such a function is called a linear fractional transformation (LFT) or a Möbius transformation. The natural domain of such functions is $\mathbb{C}_\infty$: If $c \neq 0$, set $w = \infty$ if $z = -d/c$, and set $w = a/c$ if $z = \infty$. Every linear fractional transformation T is a continuous bijection $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Since $\frac{dT}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0$ (avoiding dividing by zero), $w = Tz$ is conformal on

$$G = \begin{cases} \mathbb{C} & \text{if} \quad c = 0 \\ \mathbb{C}\backslash\{-\frac{d}{c}\} & \text{if} \quad c \neq 0 \end{cases}$$

Theorem 9.2.2. Every linear fractional transformation is a composition of the four elementary transformations: translation, rotation, dilation and inversion.

Proof. If $c = 0$, let $s_1 = \frac{a}{d}z$, $s_2 = z + \frac{b}{d}$. Then $\frac{az+b}{cz+d} = s_2 \circ s_1$. If $c \neq 0$, let $s_1 = z + \frac{d}{c}$, $s_2 = \frac{1}{z}$, $s_3 = \frac{bc-ad}{c^2}z$, and $s_4 = z + \frac{a}{c}$. Then $\frac{az+b}{cz+d} = s_4 \circ s_3 \circ s_2 \circ s_1$. $\square$

Theorem 9.2.3. Let z_2, z_3, z_4 be distinct points in $\mathbb{C}_\infty$. Then for any other distinct points w_2, w_3, w_4 in $\mathbb{C}_\infty$, there exists a unique linear fractional transformation T such that $T(z_2) = w_2$, $T(z_3) = w_3$, and $T(z_4) = w_4$.

Proof. Without loss of generality, take $w_2 = 1$, $w_3 = 0$, and $w_4 = \infty$.
If z_2, z_3, z_4 are finite, let $Tz = \left(\frac{z-z_3}{z-z_4}\right)\left(\frac{z_2-z_3}{z_2-z_4}\right)^{-1}$.
If $z_2 = \infty$, take $Tz = \frac{z-z_3}{z-z_4}$.
If $z_3 = \infty$, take $Tz = \frac{z_2-z_4}{z-z_4}$.
If $z_4 = \infty$, take $Tz = \frac{z-z_3}{z_2-z_4}$.
Uniqueness: Suppose the two linear fractional transformations S and T both satisfy the mappings $z_2 \mapsto 1$, $z_3 \mapsto 0$, $z_4 \mapsto \infty$. Then the linear fractional transformation $ST^{-1} = \frac{az+b}{cz+d}$,

$ad-bd \neq 0$, fixes the points $1, 0, \infty$. If $c \neq 0$, $ST^{-1}(\infty) = \frac{a}{c} \neq \infty$, a contradiction, so $c = 0$. Now $d \neq 0$ since $ad - bc \neq 0$, but $ST^{-1}(0) = \frac{b}{d} = 0$, so $b = 0$. Finally $ST^{-1}(1) = \frac{a}{d} = 1$. Conclude $ST^{-1} = \mathrm{id}$, i.e., $S = T$. ☐

Definition 9.2.4. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$, with z_2, z_3, z_4 distinct. Define the cross ratio (z_1, z_2, z_3, z_4) to be the image of z_1 under the unique linear fractional transformation satisfying the mappings $z_2 \mapsto 1$, $z_3 \mapsto 0$, and $z_4 \mapsto \infty$.

Theorem 9.2.5. Let T be a linear fractional transformation, z_2, z_3, z_4 distinct points in $\mathbb{C}_\infty$. Then for any $z_1 \in \mathbb{C}_\infty$, $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

Proof. Write $Sz_1 = (z_1, z_2, z_3, z_4)$. Then ST^{-1} satisfies the mappings $Tz_2 \mapsto 1$, $Tz_3 \mapsto 0$, $Tz_4 \mapsto \infty$. Thus $(Tz_1, Tz_2, Tz_3, Tz_4) = (ST^{-1})(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4)$. ☐

Corollary 9.2.6. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points, and let $w_2, w_3, w_4 \in \mathbb{C}_\infty$ be any other distinct points. The linear fractional transformation $w = Tz$ satisfying the maps $z_2 \mapsto w_2$, $z_3 \mapsto w_3$, $z_4 \mapsto w_4$ is given by the equation $(w, w_2, w_3, w_4) = (z, z_2, z_3, z_4)$.

Definition 9.2.7. A generalized circle in $\mathbb{C}_\infty$ is either a circle in $\mathbb{C}$ or a line together with the point at infinity in $\mathbb{C}_\infty$.

Theorem 9.2.8. Let $w = Tz = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, be a linear fractional transformation. Then the set $\Gamma = \{z \in \mathbb{C}_\infty : Tz \in \mathbb{R}_\infty\}$ is a generalized circle.

Proof. Suppose $z \in \Gamma$. Then $\frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$, i.e., $\frac{a\bar{c}-c\bar{a}}{2i}|z|^2 + \frac{a\bar{d}-b\bar{c}}{2i}z + \frac{b\bar{c}-\bar{a}d}{2i}\bar{z} + \frac{b\bar{d}-d\bar{b}}{2i} = 0$. Then $\frac{a\bar{c}-c\bar{a}}{2i}|z|^2 + 2\operatorname{Re}\left(\frac{a\bar{d}-b\bar{c}}{2i}z\right) + \frac{b\bar{d}-\bar{b}d}{2i} = 0$. This equation has the form $\alpha(x^2+y^2)+\beta x+\gamma y+\delta = 0$, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. If $\alpha \neq 0$, we have the equation of a circle. If $\alpha = 0$, we have the equation of a line. ☐

Corollary 9.2.9. Any three distinct points $z_2, z_3, z_4 \in \mathbb{C}_\infty$ determine a generalized circle $\Gamma = \{z \in \mathbb{C}_\infty : (z, z_2, z_3, z_4) \in \mathbb{R}_\infty\}$.

Proof. By the above theorem, Γ is a generalized circle. Let $Tz = (z, z_2, z_3, z_4)$. Then $Tz_2 = 1$, $Tz_3 = 0$, and $Tz_4 = \infty$, so $z_2, z_3, z_4 \in \Gamma$. ☐

Remark. Four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ lie on the same generlized circle if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$.

Theorem 9.2.10.

1. Let T be a linear fractional transformation, and let Γ be a generalized circle. Then $T\Gamma$ is a generalized circle.

2. If Γ_1, Γ_2 are generalized circles, then in infinitely many ways we can find a linear fractional transformation T such that $T\Gamma_1 = T\Gamma_2$.

Proof.

1. Choose $z_2, z_3, z_4 \in \Gamma$. By Corollary 4.14, $\Gamma = \{z \in \mathbb{C}_\infty : (z, z_2, z_3, z_4) \in \mathbb{R}_\infty\}$. By Theorem 4.10,

$$\begin{aligned} T\Gamma &= \{Tz \in \mathbb{C}_\infty : (z, z_2, z_3, z_4) \in \mathbb{R}_\infty\} \\ &= \{Tz \in \mathbb{C}_\infty : (Tz, Tz_2, Tz_3, Tz_4) \in \mathbb{R}_\infty\} \\ &= \{w \in \mathbb{C}_\infty : (w, Tz_2, Tz_3, Tz_4) \in \mathbb{R}_\infty\} \end{aligned}$$

which is a generalized circle by 4.13.

2. In infinitely many ways, choose distinct points $w_2, w_3, w_4 \in \Gamma_2$. Define $w = Tz$ by $(w, w_2, w_3, w_4) = (z, z_2, z_3, z_4)$. By the above argument, $T\Gamma_1 = \Gamma_2$. □

Definition 9.2.11. Let Γ be a generalized circle containing the distinct points z_2, z_3, z_4. Call z_1, z_1^* symmetric with respect to Γ if $(z_1^*, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}$.

Lemma 9.2.12. The above definition is independent of the choice of $z_2, z_3, z_4 \in \Gamma$.

Proof. Consider the points $\widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4 \in \Gamma$. Assume $(z_1^*, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}$. Set $Tz = (z, z_2, z_3, z_4)$, $\widetilde{T}z = (z, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4)$. So $Tz_1^* = \overline{Tz_1}$. Set $S = \widetilde{T}T^{-1}$. Each of $T, \widetilde{T}$ map Γ to $\mathbb{R}_\infty$. So $\mathbb{R}_\infty \xrightarrow{T^{-1}} \Gamma \xrightarrow{\widetilde{T}} \mathbb{R}_\infty$, i.e., $S(\mathbb{R}_\infty) = \mathbb{R}_\infty$. By Conway Page 55 #8, $Sz = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$. So $S\overline{z} = \overline{Sz}$. Thus $\widetilde{T}z_1^* = \widetilde{T}T^{-1}Tz_1^* = S(Tz_1^*) = S(\overline{Tz_1}) = \overline{S(Tz_1)} = \overline{\widetilde{T}T^{-1}Tz_1} = \overline{\widetilde{T}z_1}$. □

Theorem 9.2.13 (Symmetry Principle)**.** Let T be a linear fractional transformation such that $T\Gamma_1 = \Gamma_2$, where Γ_1, Γ_2 are generalized circles. If z_1, z_1^* are symmetric with respect to Γ_1, then Tz_1, Tz_1^* are symmetric with respect to Γ_2.

Proof. Choose distinct points $z_2, z_3, z_4 \in \Gamma_1$. Now $(Tz_1^*, Tz_2, Tz_3, Tz_4) \overset{(4.10)}{=} (z_1^*, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)} \overset{(4.10)}{=} \overline{(Tz_1, Tz_2, Tz_3, Tz_4)}$. Thus Tz_1^* and Tz_1 are symmetric with respect to Γ_2. □

Remark (Meaning of symmetry with respect to a line Γ)**.** Let $z_2, z_3, z_4 \in \Gamma$, $z_4 = \infty$. Then $\frac{z_1^*-z_3}{z_2-z_3} = \frac{\overline{z_1}-\overline{z_3}}{\overline{z_2}-\overline{z_3}}$, so $|z_1^* - z_3| = |z_1 - z_3|$. Since $\operatorname{Im}\left(\frac{z_1^*-z_3}{z_2-z_3}\right) = \operatorname{Im}\left(\frac{\overline{z_1}-\overline{z_3}}{\overline{z_2}-\overline{z_3}}\right) = -\operatorname{Im}\left(\frac{z_1-z_3}{z_2-z_3}\right)$, we conclude that z_1 and z_1^* are on opposite sides of the line.

Remark (Meaning of symmetry with respect to a cirlce)**.** Suppose z_1 and z_1^* are symmetric with respect to the circle $\Gamma = \{z : |z-a| = R\}$. Then

$$\begin{aligned}
(z_1^*, z_2, z_3, z_4) &= \overline{(z_1, z_2, z_3, z_4)} \\
&= \overline{(z_1 - a, z_2 - a, z_3 - a, z_4 - a)} \quad \text{translation} \\
&= (\overline{z_1} - \overline{a}, \overline{z_2} - \overline{a}, \overline{z_3} - \overline{a}, \overline{z_4} - \overline{a}) \\
&= \left(\overline{z_1} - \overline{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a}\right) \\
&= \left(\frac{R^2}{\overline{z_1} - \overline{a}}, z_2 - a, z_3 - a, z_4 - a\right) \quad \text{invert and multiply by } R^2 \\
&= \left(\frac{R^2}{\overline{z_1} - \overline{a}} + a, z_2, z_3, z_4\right) \quad \text{translation}
\end{aligned}$$

Thus $z_1^* = \frac{R^2}{\overline{z_1}-\overline{a}} + a$, or $(z_1^* - a)(\overline{z_1} - \overline{a}) = R^2 \Rightarrow |z_1^* - a||z_1 - a| = R^2$. Also, $\frac{z_1^*-a}{z_1-a} = \frac{R^2}{|z-a|^2} > 0$, so $z_1^* - a$ and $z_1 - a$ have the same argument $\Rightarrow$ $z_1^* - a$ and $z_1 - a$ are on the same ray.

Example 9.2.14. Find all linear fractional transformations T that map the unit circle onto itself. Let $T1 = e^{i\theta}$. Suppose $T\alpha = 0$. If $\alpha \neq \infty$, by symmetry we have $T(1/\overline{\alpha}) = \infty$. Then $w = Tz$ is determined by the equation $(w, e^{i\theta}, 0, \infty) = (z, 1, \alpha, 1/\overline{\alpha})$, i.e., $w = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$. If $\alpha = \infty$, then $w = Tz$ is determined by the equation $(w, e^{i\theta}, 0, \infty) = (z, 1, \infty, 0)$, i.e., $w = e^{i\theta}\frac{1}{z}$.

Definition 9.2.15. The orientation of a generalized circle Γ is an ordered triple (z_2, z_3, z_4), $z_2, z_3, z_4 \in \Gamma$. (So $\Gamma = \{z : \operatorname{Im}((z, z_2, z_3, z_4)) = 0\}$.) Define the right side R of Γ by $R = \{z : \operatorname{Im}((z, z_2, z_3, z_4)) > 0\}$. Define the left side L of Γ by $L = \{z : \operatorname{Im}((z, z_2, z_3, z_4)) < 0\}$.

Theorem 9.2.16 (Orientation Principle)**.** A linear fractional transformation T will map the left (right) side of the generalized circle Γ to the left (right) side of the generalized circle $T\Gamma$.

Proof. Follows from Theorem 4.10. □

Example 9.2.17. Let $D = \{z : |z| < 1\}$, and let $\mathbb{C}_+ = \{z : \operatorname{Im}(z) > 0\}$. (a) Find all linear fractional transformations T such that $TD = D$. All such LFTs are given by the formula above in 4.19 with $\theta \in \mathbb{R}$ and $|\alpha| < 1$. (b) Map $\{z : |z| < R, \operatorname{Im}(z) > 0\}$ onto D via rational functions. (c) The Joukowski function $w = \frac{1}{2}(z + \frac{1}{z})$ has derivative $\frac{dw}{dz} = \frac{1}{2}(1 - z^{-2}) \neq 0$ if $z \neq \pm 1$. The Joukowski function maps circles to ellipses, and rays to hyperbolas. (d) Mapping properties of e^z and $\log z$.

Definition 9.2.18 (Riemann Surfaces)**.** Purpose: to simplify the geometric view of a one-to-many or a many-to-one function.

Chapter 10

Integration Theory

10.1 Basic Concepts

Definition 10.1.1. If $u, v : [a, b] \to \mathbb{R}$ are continuous, define

$$\int_a^b u(t) + iv(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Let $\varphi(t) = u(t) + iv(t)$. This integral inherits various properties from the real integral by virtue of linearity. In particular, $\left|\int_a^b \varphi(t)dt\right| \leq \int_a^b |\varphi(t)|dt$.

Proof. We have that $\left|\int_a^b \varphi(t)dt\right| = e^{i\theta}\int_a^b \varphi(t)dt$ for some $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\left|\int_a^b \varphi(t)dt\right| &= e^{i\theta}\int_a^b \varphi(t)dt \\ &= \int_a^b e^{i\theta}\varphi(t)dt \\ &= \int_a^b \mathrm{Re}\left(e^{i\theta}\varphi(t)\right)dt \leq \int_a^b |e^{i\theta}\varphi(t)|dt \\ &= \int_a^b |\varphi(t)|dt\end{aligned}$$

□

Definition 10.1.2 (Definition of a Curve)**.** By a smooth arc we mean a parametric curve $\gamma(t) = x(t) + iy(t)$, $a \leq t \leq b$, such that $\gamma(t)$ is continuously differentiable on $[a, b]$ and $\gamma'(t) \neq 0$ on $[a, b]$. By a curve or path we mean a finite set of such arcs $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ such that the initial point of γ_i is the final point of γ_{i-1}, $2 \leq i \leq n$. By a closed curve, we mean a curve with idential initial and end points. We define the opposite curve γ_{op} or $-\gamma$ by $\gamma_{op} := \gamma(a+b-t)$.

Definition 10.1.3 (Change of Parameter, Length of a Curve, Parametrization by Arc Length)**.** Given a curve $\gamma : [a,b] \to \mathbb{C}$, we say $\widetilde{\gamma} : [c,d] \to \mathbb{C}$ is obtained from γ via a change of parameter if there exists a continuously differentiable function $\varphi : [c,d] \to [a,b]$ with $\varphi'(s) > 0 \, \forall \, s \in [c,d]$ such that $\widetilde{\gamma} = \gamma \circ \varphi$. If this is the case, we call γ and $\widetilde{\gamma}$ equivalent curves. We define the length $L(\gamma)$ of the curve γ by $L(\gamma) := \int_a^b |\gamma'(t)|dt$. Given a curve γ, set $s = s(t) = \int_a^t |\gamma'(u)|du$. Let $t = \varphi(s)$ be the inverse function. We call $\gamma(\varphi(s))$ the arc length parametrization of γ, $0 \le s \le L(\gamma)$.

Definition 10.1.4 (Line Integrals)**.** Let $\gamma : I \to \mathbb{C}$ be a curve, $\operatorname{image}(\gamma) \subset G$ for some region G, and $f : G \to \mathbb{C}$ continuous. Define the line integral of f along γ, denoted by $\int_\gamma f(z)dz$ or more simply $\int_\gamma f$ by $\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$.

Theorem 10.1.5 (Invariance Under Change of Parameter)**.** If $\gamma, \widetilde{\gamma} : [a,b] \to G$ are equivalent curves, and $f : G \to \mathbb{C}$ is continuous, then $\int_\gamma f = \int_{\widetilde{\gamma}} f$.

Proof. Write $\widetilde{\gamma} = \gamma \circ \varphi$ for some continuously differentiable function $\varphi : [c,d] \to [a,b]$. Then $\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_c^d f(\gamma(\varphi(s)))\gamma'(\varphi(s))\varphi'(s)ds = \int_c^d f(\widetilde{\gamma}(s))\widetilde{\gamma}'(s)ds = \int_{\widetilde{\gamma}} f$. ☐

Theorem 10.1.6. Let γ be a curve on a region G, $f : G \to \mathbb{C}$ continuous. If $f = F'$, for some analytic function F on G, then $\int_\gamma f(z)dz = F(w_2) - F(w_1)$, where w_1, w_2 are the initial and end points of γ.

Proof. Write $\gamma(t) = x(t) + iy(t)$, $a \le t \le b$. Assume γ is continuously differentiable. Then $\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a))$, with the last equality true by the Fundamental Theorem of Calculus after splitting F into its real and imaginary parts. ☐

Definition 10.1.7. Let γ be a curve on a region G, $f : G \to \mathbb{C}$ continuous. Then we make the following definitions:

(i) $\int_\gamma f(z)|dz| := \int_\gamma f(z)ds = \int_a^b f(\gamma(t))|\gamma'(t)|dt$

(ii) $\int_\gamma f(z)\overline{dz} := \overline{\int_\gamma \overline{f(z)}dz} = \int_a^b f(\gamma(t))\overline{\gamma'(t)}dt$

(iii) $\int_\gamma f(z)dx := \frac{1}{2}\left(\int_\gamma f(z)dz + \int_\gamma f(z)\overline{dz}\right) = \int_a^b f(\gamma(t))\,\mathrm{Re}\,(\gamma'(t))\,dt$

(iv) $\int_\gamma f(z)dy := \frac{1}{2i}\left(\int_\gamma f(z)dz - \int_\gamma f(z)\overline{dz}\right) = \int_a^b f(\gamma(t))\,\mathrm{Im}\,(\gamma'(t))\,dt$

(v) $\int_\gamma (p\,dx + q\,dy) := \int_\gamma p\,dx + \int_\gamma q\,dy$

10.2 Properties of the Line Integral

Theorem 10.2.1. Let γ be a curve on a region G, $f : G \to \mathbb{C}$ continuous. Then the following hold:

1. $\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$
2. $\left|\int_{\gamma} f(z)dz\right| \leq \int_{\gamma} |f(z)||dz|$
3. If $|f| \leq M$ on γ and $L = L(\gamma)$, then $\left|\int_{\gamma} f(z)dz\right| \leq ML$.

Proof. Suppose the domain of $\gamma(t)$ is some closed interveral $[a, b]$. Then

$$\begin{aligned}
\int_{-\gamma} f(z)dz &= \int_a^b f(\gamma(a+b-t))(\gamma(a+b-t))'dt \\
&= -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)dt \\
&= \int_b^a f(\gamma(u))\gamma'(u)du \\
&= -\int_a^b f(\gamma(u))\gamma'(u)du = \int_{\gamma} f(z)dz \\
\left|\int_{\gamma} f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \\
&\leq \int_a^b |f(\gamma(t)\gamma'(t)|\, dt = \int_{\gamma} |f(z)||dz| \\
\left|\int_{\gamma} f(z)dz\right| &\leq \int_{\gamma} |f(z)||dz| \\
&\leq M\int_{\gamma} |dz| = ML
\end{aligned}$$

□

Theorem 10.2.2. Let $p, q : G \to \mathbb{C}$ be continuous on the region G. Then the following are equivalent:

1. $\int_{\gamma}(p\,dx + q\,dy)$ depends only on the end points of any curve γ in G.
2. $\int_{\gamma}(p\,dx + q\,dy) = 0$ for any closed curve γ in G.
3. There is a function u on G such that $p = u_x$ and $q = u_y$.

Proof.
(i) $\Leftrightarrow$ (ii) Follows from 5.8(1).
(iii) $\Rightarrow$ (i) Consider a curve $\gamma(t) = x(t) + iy(t)$. Then

$$\begin{aligned} \int_\gamma (p\,dx + q\,dy) &= \int_a^b u_x(x(t), y(t))x'(t) + u_y(x(t), y(t)y'(t)dt \\ &= \int_a^b \frac{d}{dt} u(x(t), y(t))dt \\ &= u(x(b), y(b)) - u(x(a), y(a)) \end{aligned}$$

where the last equality holds by the Fundamental Theorem of Calculus after splitting u into its real and imaginary parts. Thus the integral depends only on the initial and end points.
(i) $\Rightarrow$ (iii) Fix $z_0 \in G$. Define $u(z) = \int_{\gamma_z}(p\,dx + q\,dy)$, where γ_z is some path from z_0 to z. Then u is well-defined by the assumption. First choose γ_z so as to approach z from the left, parallel to the real axis. Then $u(z) = \text{constant} + \int_c^x [p(t+iy)\cdot 1 + q(t+iy)\cdot 0]dt$. (We have taken the path to be $t+iy$, $c \le t \le x$ for some c and x.) Then $u_x(z) = p(x+iy) = p(z)$. Next choose γ_z to approach z from below, parallel to the imaginary axis. Then $u(z) = \text{constant} + \int_c^y [p(x+it)\cdot 0 + q(x+it)\cdot 1]dt$, and $u_y(z) = q(x+iy) = q(z)$. □

Corollary 10.2.3. Let G be a region, $f : G \to \mathbb{C}$ a continuous function. Then the following are equivalent:

1. $\int_\gamma f(z)dz$ depends only on the endpoints of γ.
2. $\int_\gamma f(z)dz = 0$ for any closed curve γ in G.
3. f has a primitive g, i.e., there exists an analytic function g on G such that $g' = f$.

Proof.
(i) $\Leftrightarrow$ (ii) As before.
(i) $\Rightarrow$ (iii) We have $\int_\gamma f(z)dz = \int_\gamma [f(z)dx + if(z)dy]$. Then by the previous theorem, there exists a function $g : G \to \mathbb{C}$ such that $g_x = f$ and $g_y = if$. Then g has continuous partial derivatives and satisfies the equation $\frac{\partial g}{\partial x} = -i\frac{\partial g}{\partial y}$. Then by 2.7(2), g is analytic and $g' = g_x = f$.
(iii) $\Rightarrow$ (i) Use Theorem 5.6. □

Theorem 10.2.4 (Rectangle Theorem)**.** Let f be analytic on a region G containing a rectangle $R : x_1 \le x \le x_2, y_1 \le y \le y_2$. Then $\int_{\partial R} f(z)dz = 0$.

Proof. Remark: This is true if $f(z) = a + bz$, since $f(z) = \frac{d}{dz}\left(az + \frac{1}{2}bz^2\right)$. Let $\epsilon > 0$. Set $R_0 = R$, $L = L(\partial R)$, d the diagonal of R. Divide R_0 into four equal rectangles R_1, R_2, R_3, R_4.

Then

$$\int_{\partial R} f(z)dz = \int_{\partial R_1} f(z)dz + \int_{\partial R_2} f(z)dz + \int_{\partial R_3} f(z)dz + \int_{\partial R_4} f(z)dz$$

At least one of the subrectangles, say R_1, must satisfy $\left|\int_{\partial R_1} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R_0} f(z)dz\right|$. Subdivide the rectangle R_1 and obtain a new subrectangle R_2 satisfying

$$\left|\int_{\partial R_2} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R_1} f(z)dz\right| \geq \frac{1}{16}\left|\int_{\partial R_0} f(z)dz\right|$$

At the m-th stage, we have $\left|\int_{\partial R_m} f(z)dz\right| \geq \frac{1}{4^m}\left|\int_{\partial R_0} f(z)dz\right|$. Let $L_m = L(\partial R_m) = L/2^m$, $d_m = d/2^m$ the diagonal of R_m. The nested rectangles $R_0 \supset R_1 \supset R_2 \supset \cdots$ intersect at a point $z^* \in G$. Thus f is differentiable at z^*. We can choose $\delta > 0$ such that (1) $D(z^*, \delta) \subset G$, and (2) $\forall\, z \in D(z^*, \delta)$, $|f(z) - f(z^*) - (z - z^*)f'(z^*)| \leq \epsilon\, |z - z^*|$. Then by the preliminary remark, $\left|\int_{\partial R_m} f(z)dz\right| = \left|\int_{\partial R_m}[f(z) - f(z^*) - (z - z^*)f'(z^*)]dz\right|$, where $a + bz = f(z^*) + (z - z*)f'(z^*)$. Now for all sufficiently large n, say $n \geq n_0$, $R_n \subset D(z^*, \delta)$. Hence for $n \geq n_0$,

$$\begin{aligned} \left|\int_{\partial R_n} f(z)dz\right| &\leq \int_{\partial R_n} |f(z) - f(z^*) - (z - z^*)f'(z^*)|\, |dz| \\ &\leq \int_{\partial R_n} \epsilon\, |z - z^*|\, |dz| \\ &\leq \epsilon d_n L_n \\ &= \epsilon \frac{dL}{4^n} \end{aligned}$$

Thus $\left|\int_{\partial R} f(z)dz\right| \leq 4^n \left|\int_{\partial R_n} f(z)dz\right| \leq \epsilon dL$. Conclude $\int_{\partial R} f(z)dz = 0$. □

Theorem 10.2.5. The rectangle theorem remains true if f is analytic on G except at perhaps one point z_0 and f is continuous at z_0.

Proof. Suppose z_0 is in the interior of R. For any small square S centered at z_0, $\int_{\partial R} f = \int_{\partial S} f$. Choose $M \geq 0$ such that $|f| \leq M$ on S. Then $\left|\int_{\partial R} f\right| = \left|\int_{\partial S} f\right| \leq ML(\partial S) \to 0$, since we can make $L(\partial S) \to 0$. □

Theorem 10.2.6 (Cauchy's Theorem for a disk)**.** If f is analytic on an open disk $\triangle$, then $\int_\gamma f = 0$ for every closed curve γ in $\triangle$.

Proof. By 5.6, it is enough to show that there exists an analytic function g on $\triangle$ such that $g' = f$. Let a be the center of the disk $\triangle$, and define $g(z) = \int_{\gamma_{a,z}} f(w)dw$, where $\gamma_{a,z}$ is the path in $\triangle$ from a to z composed first of a straight line segment parallel to the real axis, and then a straight line segment parallel to the imaginary axis. Fix $z_0 \in \triangle$, $r > 0$ such that $D(z_0, r) \subset \triangle$. Consider $h \in \mathbb{C}$ such that $0 < |h| < r$. Then

$$g(z_0 + h) - g(z_0) = \int_{\gamma_{a,z_0+h}} f(w)dw - \int_{\gamma_{a,z_0}} f(w)dw = \int_{\gamma_{z_0,z_0+h}} f(w)dw$$

where the last equality is true by the Rectangle Theorem, and γ_{z_0,z_0+h} is a path defined similarly as above for $\gamma_{a,z}$. Observe that $\int_{\gamma_{z_0,z_0+h}} 1\, dw = w|_{z_0}^{z_0+h} = h$. Then

$$\frac{g(z_0 + h) - g(z_0)}{h} - f(z_0) = \frac{1}{h}\int_{\gamma_{z_0,z_0+h}} [f(w) - f(z_0)]dw$$

Let $\epsilon > 0$. Since f is continuous at z_0, there exists $\delta > 0$, $0 < \delta < r$, such that $|f(w)-f(z_0)| < \epsilon/2$ for $|w - z_0| < \delta$. Then if $0 < |h| < \delta$,

$$\left|\frac{g(z_0 + h) - g(z_0)}{h} - f(z_0)\right| \leq \frac{1}{|h|}\frac{\epsilon}{2}L(\gamma_{z_0,z_0+h}) \leq \frac{1}{|h|}\frac{\epsilon}{2}2|h| = \epsilon$$

Conclude $g'(z) = f(z)$. □

Theorem 10.2.7. Cauchy's Theorem for a disk remains true if f if analytic except at possibly one point z_0, and f is continuous at z_0.

Proof. Same as above, except use the version of the rectangle theorem that permits f to be non-analytic at a single point where it is continuous. □

Example 10.2.8. Let $a \in \mathbb{C}$, $r > 0$. Let $\gamma_r(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Then $\int_{\gamma_r} \frac{1}{z-a}dz = 2\pi i$.

Definition 10.2.9 (Elementary Deformations)**.** Let f be analytic on G, and let γ_1 be a curve in G. We say the curve γ_2 differs from γ_1 via an elementary deformation if there exists an open disk $\triangle \subset G$ such that $\gamma_1 = \gamma_2$ on $G\backslash\triangle$, and inside $\triangle$, γ_2 is obtained from γ_1 by replacing an arc of γ_1 contained in $\triangle$ with any other arc contained in $\triangle$. We say the curve γ_2 is a deformation of the curve γ_1, written $\gamma_1 \sim \gamma_2$, if γ_2 is obtained from γ_1 via a finite number of elementary transformations.

Theorem (Deformation Principle)**.** Let f be analytic on G, and let γ_1, γ_2 be two curves in G. If $\gamma_1 \sim \gamma_2$, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

Proof. By Cauchy's Theorem, the theorem is true for elementary deformations. The theorem then follows since γ_2 is obtained from γ_1 via a finite number of elementary deformations. □

Theorem 10.2.10 (Cauchy's Theorem for Simply Connected Regions)**.** We say a region G is simply connected if every closed curve γ in G can be deformed to a constant loop in G. If

f is analytic on a simply connected region G, then $\int_\gamma f(z)dz = 0$ for every closed curve γ in G.

Theorem. Let $G \subsetneq \mathbb{C}$ be a region. Then the following are equivalent:

1. G is simply connected.
2. $\mathbb{C}_\infty \backslash G$ is connected
3. G is homeomorphic to an open disk.

Proof. Conway Page 202. $\square$

Definition 10.2.11 (Winding Number of a Closed Curve About a Point)**.** If γ is a closed curve with $w \notin \gamma$, define $n(\gamma, w) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$ the winding number of γ about w.

Theorem 10.2.12 (Properties of the Winding Number)**.** The winding number $n(\gamma, w)$ of a closed curve γ is integer-valued, is constant on the components of $\mathbb{C}\backslash\gamma$, and is equal to zero on the unbounded component of $\mathbb{C}\backslash\gamma$.

Proof. Let $\gamma(u) = z(u) : [a, b] \to \mathbb{C}$. Fix $w \notin \gamma$. Set $h(t) = \int_a^t \frac{z'(u)}{z(u)-w}du$. Then $h(t)$ is continuous, $h(b) = \int_\gamma \frac{dz}{z-w}$, and $h'(t) = \frac{z'(t)}{z(t)-w}$ at all but finitely many points. Then $\frac{d}{dt}\left[e^{-h(t)}(z(t) - w)\right] = 0$ at all but finitely many points. By the Mean Value Theorem (applied separately to the real and imaginary parts), $e^{-h(t)}(z(t) - w) =$ constant. Hence $e^{-h(a)}(z(a)-w) = e^{-h(b)}(z(b)-w) \Rightarrow z(a)-w = e^{-h(b)}(z(b)-w)$. Since $z(a) = z(b)$, $e^{h(b)} = 1$. So $h(b) = \int_\gamma \frac{dz}{z-2} = 2\pi ik$ for some $k \in \mathbb{Z}$. Given that $\varphi(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$ is differentiable, it is continuous and integer-valued. Thus it must be constant on the connected components of G. By the ML formula, $\lim_{|w|\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w} = 0$, so $n(\gamma, w) = 0$ on the unbounded component. $\square$

10.3 Connection to Analytic Functions

Lemma 10.3.1 (Differentiation of Integrals)**.** For a curve γ (closed or not) and a continuous function f on γ, $F(z) = \int_\gamma \frac{f(\zeta)}{\zeta-z}d\zeta$ is infinitely differentiable, and $F^{(n)}(z) = n!\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$.

Proof. Set $F(z) = \int_\gamma \frac{f(\zeta)}{\zeta-z}d\zeta$. Then for $z \notin \gamma$,

$$\begin{aligned}
\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} &= \lim_{h\to 0}\frac{1}{h}\int_\gamma \frac{f(\zeta)}{\zeta-(z+h)} - \frac{f(\zeta)}{\zeta-z}\,d\zeta \\
&= \lim_{h\to 0}\int_\gamma \frac{1}{h}\frac{f(\zeta)(\zeta-z) - f(\zeta)(\zeta-(z+h)}{(\zeta-z)(\zeta-(z+h))}\,d\zeta \\
&= \lim_{h\to 0}\int_\gamma \frac{f(\zeta)}{(\zeta-z)((\zeta-z)-h)}\,d\zeta
\end{aligned}$$

Let $M = \max\{|f(\zeta)| : \zeta \in \gamma\}$, and let $2m = \min\{|\zeta - z| : \zeta \in \gamma\}$. Then for $|h| < m$, $|(\zeta - z)((\zeta - z) - h)| \geq (2m)(2m - |h|) \geq (2m)(m) \geq m^2$, and

$$\left|\frac{f(\zeta)}{(\zeta - z)((\zeta - z) + h)}\right| \leq \frac{M}{m^2}$$

So by the Dominated Convergence Theorem with dominating function M/m^2,

$$\begin{aligned}\lim_{h\to 0}\int_\gamma \frac{f(\zeta)}{(\zeta - z)((\zeta - z) - h)}\,d\zeta &= \int_\gamma \lim_{h\to 0}\frac{f(\zeta)}{(\zeta - z)((\zeta - z) - h)}\,d\zeta \\ &= \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta\end{aligned}$$

Now assume that the formula is valid up to order $n - 1$. We have that

$$\lim_{h\to 0}\frac{F^{(n-1)}(z + h) - F^{(n-1)}(z)}{h} = \lim_{h\to 0}(n-1)!\int_\gamma -\frac{f(\zeta)}{h}\frac{((\zeta - z) - h)^n - (\zeta - z)^n}{(\zeta - z)^n((\zeta - z) - h)^n}\,d\zeta$$

Note that

$$\begin{aligned}\frac{1}{h}\left[((\zeta - z) - h)^n - (\zeta - z)^n\right] &= \left(\sum_{k=0}^{n}\binom{n}{k}(\zeta - z)^k(-1)^{n-k}(h)^{n-k-1}\right) - \frac{(\zeta - z)^n}{h} \\ &= \sum_{k=0}^{n-1}\binom{n}{k}(\zeta - z)^k(-1)^{n-k}(h)^{n-k-1}\end{aligned}$$

Let $M' = \max\{|\zeta - z| : \zeta \in \gamma\}$. Then for $|h| < m$,

$$C := \sum_{k=0}^{n-1}\binom{n}{k}(M')^k m^{n-k-1} \geq \left|\sum_{k=0}^{n-1}\binom{n}{k}(\zeta - z)^k(-1)^{n-k}(h)^{n-k-1}\right|$$

So by the Dominated Convergence Theorem with dominating function $(MC)/(m^{2n})$,

$$\begin{aligned}\lim_{h\to 0}\int_\gamma -\frac{f(\zeta)}{h}\frac{((\zeta - z) - h)^n - (\zeta - z)^n}{(\zeta - z)^n((\zeta - z) - h)^n}\,d\zeta &= \int_\gamma \lim_{h\to 0} -\frac{f(\zeta)}{h}\frac{((\zeta - z) - h)^n - (\zeta - z)^n}{(\zeta - z)^n((\zeta - z) - h)^n}\,d\zeta \\ &= \int_\gamma f(\zeta)\frac{n(\zeta - z)^{n-1}}{(\zeta - z)^{2n}}\,d\zeta \\ &= \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta\end{aligned}$$

and the result follows. □

Theorem 10.3.2 (Cauchy's Integral Formula)**.** Let f be analytic on a region containing the circle $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, and its interior. If $|z-a| < r$, then $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ and $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}} d\zeta$. In particular, f is infinitely differentiable.

Proof. Fix $z_0 \in D(a,r)$. Set

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta)-f(z_0)}{\zeta - z_0} & \text{if} \quad \zeta \neq z_0 \\ f'(z_0) & \text{if} \quad \zeta = z \end{cases}$$

Then g is analytic everywhere, except possibly at z_0 and is continuous at z_0. By 5.14, $\int_\gamma g(\zeta)\, d\zeta = 0$. That is, $\int_\gamma \frac{f(\zeta)-f(z_0)}{\zeta - z_0}\, d\zeta = 0$. Then

$$\begin{aligned} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} d\zeta &= f(z_0)\int_\gamma \frac{d\zeta}{\zeta - z_0} \\ &= f(z_0)\cdot 2\pi i \cdot n(\gamma, z_0) \\ &= 2\pi i \cdot f(z_0) \end{aligned}$$

Thus $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0} d\zeta$. The formula for $f^{(n)}(z)$ then follows from 5.20. □

Corollary 10.3.3. An analytic function is infinitely differentiable. Harmonic functions have continuous partial derivatives of all orders.

Theorem 10.3.4 (Morera's Theorem)**.** Let f be a continuous function on a region G. Assume $\int_R f(z)dz = 0$ for every closed rectangle $R \subset G$. Then f is analytic on G.

Proof. Take G to be an open disk. The property $\int_R f(z)dz = 0$ for every closed rectangle $R \subset G$ is all that is used in the proof of 5.13 Cauchy's Theorem for a Disk. This property was used to construct an analytic primitive g for f. Now by 5.22, $f = g'$ is differentiable, hence analytic. □

Corollary 10.3.5. Let f be analytic on a region G except possibly at isolated points, and assume that f is continuous at those isolated points. Then f is analytic on all of G.

Proof. Same as above except appeal to the proof of the modified version of Cauchy's Theorem which permits isolated points of continuity but not necessarily differentiability. □

Theorem 10.3.6 (Cauchy's Formula for a Simply Connected Region)**.** Let f be analytic on a simply connected region G. Let γ be any closed curve in G. Then for any $z \in G\backslash\gamma$, $n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$.

Proof. Fix $z_0 \in G \backslash \gamma$. Set

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z_0)}{\zeta - z_0} & \text{if} \quad \zeta \neq z_0 \\ f'(z_0) & \text{if} \quad \zeta = z_0 \end{cases}$$

Then g is analytic on G except possibly at the point $\zeta = z_0$, where it is continuous. Then by the modified version of Morera's Theorem, g is analyic on all of G. Then by 5.17, $\int_\gamma g(\zeta) d\zeta = 0$. So

$$\int_\gamma \frac{f(\zeta)}{\zeta - z_0} \, d\zeta = f(z_0) \int_\gamma \frac{1}{\zeta - z_0} \, d\zeta = f(z_0) \cdot 2\pi i \cdot n(\gamma, z_0)$$

That is, $n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$. ☐

Example 10.3.7 (Example of Morera's Theorem)**.** Prove that $f(z) = \int_0^\infty e^{-zt-t^4} dt$ is an entire function.

Chapter 11

Consequences of Cauchy's Theorem

11.1 Analytic Functions as Power Series

Remark. If f, f_n are continuous on a curve γ and $f_n \to f$ uniformly on γ, then $\int_\gamma f_n \to \int_\gamma f$. For if $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n - f| < \epsilon$, $\forall\, n \geq \mathbb{N}$. Then for $n > N$, $\left|\int_\gamma f_n - \int_\gamma f\right| \leq \int_\gamma |f_n - f| \leq \epsilon L(\gamma)$.

Remark. If $f(\zeta) = \sum_1^\infty f_n(\zeta)$ converges uniformly on γ, then $\int_\gamma f = \sum_1^\infty \int_\gamma f_n$.

Theorem 11.1.1 (Power Series Representation of Analytic Functions)**.** Let f be analytic on a region G, and let $a \in G$. Then f has a representation $f(z) = \sum_{n=0}^\infty c_n (z-a)^n$ valid in the largest disk centered at a fully contained in G. Moreover, for $n \geq 0$, $c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}} = \frac{1}{n!} f^{(n)}(a)$, where γ is a cirlce centered at a contained in G.

Proof. Suppose $\overline{D(a,r)} \subset G$ and $|z-a| < r_1 < r$. Let $\gamma_r(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}
f(z) &= \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)\left[1 - \frac{z-a}{\zeta - a}\right]} d\zeta \\
&= \frac{1}{2\pi i} \int_{\gamma_r} \left(\sum_{n=0}^\infty \frac{(z-a)^n}{(\zeta - a)^{n+1}} \right) f(\zeta) d\zeta \\
&= \sum_{n=0}^\infty \frac{1}{2\pi i} \int_{\gamma_r} \frac{(z-a)^n}{(\zeta - a)^{n+1}} f(\zeta) d\zeta \\
&= \sum_{n=0}^\infty c_n (z-a)^n
\end{aligned}$$

where $c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}$, and the interchanging of the order of integration and summation is justified by the uniform convergence of the series on γ. $\square$

Theorem 11.1.2 (Uniqueness Theorem)**.** Let f be analytic on a region G. If $f(z) = 0$ on a set with a limit point in G, then $f \equiv 0$ on G.

Proof. Suppose that for $n \geq 0$, $f(z_n) = 0$ where $\{z_n\}_{n=1}^{\infty}$ are distinct points in G such that $z_n \to a$, $a \in G$. Write $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ in $D(a,r) \subset G$. Then $c_0 = f(a) = \lim_{n\to\infty} f(z_n) = 0$. Let $f_1(z) = \sum_{n=0}^{\infty} c_{n+1}(z-a)^n$. Then $f_1(z_n) = \frac{f(z_n)}{z_n - a} = 0$. So $c_1 = f_1(a) = \lim_{n\to\infty} f_1(z_n) = 0$. Continuing in this way, $c_n = 0, \forall\, n \geq 0$. So $f(z) \equiv 0$ on $D(a,r)$. Let b be another point in G. Since G is path-connected, there exists a path $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = a$ and $\gamma(1) = b$. We can cover γ by a finite number of open discs. By the above argument, $f \equiv 0$ on each of those open discs. Hence $f(b) = 0$. $\square$

Example 11.1.3.

1. With the principal branch of the logarithm and $|z| < 1$, $\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}z^n$.
2. Define $(1+z)^p := e^{p\log(1+z)}$, with the principal branch of the logarithm and any $p \in \mathbb{C}$. Then $(1+z)^p = \sum_{n=0}^{\infty}\binom{p}{n}z^n$, where

$$\binom{p}{n} = \begin{cases} 1 & \text{if} \quad n = 0 \\ \dfrac{p(p-1)\cdots(p-n+1)}{n!} & \text{if} \quad n \geq 1 \end{cases}$$

By induction, $\dfrac{d^n}{dz^n}(1+z)^p = p(p-1)\cdots(p-n+1)(1+z)^p(1+z)^{-n}$.

Theorem 11.1.4 (Cauchy's Estimate)**.** Let f be analytic on a region G. Suppose $D(a,R) \subset G$. If $|f| \leq M$ on $\gamma(t) = a + Re^{it}$, $0 \leq t \leq 2\pi$, then $|f^{(n)}(a)| \leq \frac{Mn!}{R^n}$, $\forall\, n \geq 0$.

Proof. By Cauchy's Integral Formula,

$$\begin{aligned} |f^{(n)}(a)| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta\right| \\ &\leq M \cdot \frac{n!}{2\pi} \cdot \frac{1}{R^{n+1}} \cdot \int_\gamma d\zeta \\ &= M \cdot \frac{n!}{2\pi} \cdot \frac{1}{R^{n+1}} \cdot 2\pi R \\ &= \frac{Mn!}{R^n} \end{aligned}$$

$\square$

Theorem 11.1.5 (Liouville's Theorem)**.** A bounded entire function is constant.

Proof. Suppose f is an entire function with $|f| \leq M$ on $\mathbb{C}$, for some $M \geq 0$. Then by Cauchy's Estimate, for any $a \in \mathbb{C}$ and $r > 0$, $|f'(a)| \leq \frac{M}{r} \to 0$ as $r \to \infty$. Conclude that $f'(a) = 0, \forall\, a \in \mathbb{C}$. Hence f is constant. $\square$

Theorem 11.1.6 (Fundamental Theorem of Algebra)**.** Every non-constant polynomial with complex coefficients has a complex root.

Proof. Suppose $p(z) = a_n z^n + \cdots + a_0$ is a non-constant polynomial of degree ≥ 1 with complex coefficients having no complex roots. Then $f(z) = \frac{1}{p(z)}$ is an entire function. Since $f(z) = \frac{1}{z^n(a_n + a_{n-1}z^{-1} + \cdots + a_0 z^{-n})} \to 0$ as $z \to \infty$, f is bounded. Then by Liouville's Theorem, f is constant, which implies that $p(z)$ is constant, a contradiction to the initial assumption. Conclude that $p(z)$ has at least one complex root. $\square$

Remark. By the division algorithm, every polynomial can be written as a product of linear factors $p(z) = c(z - c_1) \cdots (z - c_n)$.

Definition 11.1.7 (Zeros, c-Points of an Analytic Function)**.** Let f be a non-constant analytic function on a region G. Call $a \in G$ a zero of f if $f(a) = 0$. Let n be the first integer such that $f^{(n)}(a) \neq 0$. Call n the order or multiplicity of the zero a. If $n = 1$, call a a simple zero of f. Let $c \in \mathbb{C}$. Call a a c-point of f if $f(a) = c$. Call a a c-point of order n if a is a zero of order n of the analytic function $f(z) - c$.

Remark (Counting Zeros)**.** Suppose f is analytic and non-constant on the region G, and $\overline{D(a, r)} \subset G$. Assume $f(z) \neq 0$ on $\gamma_r = a + re^{it}$, $0 \leq t \leq 2\pi$. Then f has only finitely many zeros in $D(a, r)$. In particular, the zeros of f are isolated. List them as $a_1, \ldots, a_n$, counted according to multiplicity. Then $f(z) = (z - a_1)(z - a_2) \cdots (z - a_n)g(z)$ for some g analytic on G and non-zero on $\overline{D(a, r)}$. Now $\frac{f'(z)}{f(z)} = \frac{1}{z - a_1} + \cdots + \frac{1}{z - a_n} + \frac{g'(z)}{g(z)}$, and $\frac{1}{2\pi i} \int_{\gamma_r} \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{n} n(\gamma_r, a_i) = n$. Given a point $c \in \mathbb{C}$, $\frac{1}{2\pi i} \int_{\gamma_r} \frac{f'(z)}{f(z) - c} dz$ is the number of c-points of f in $D(a, r)$ counted according to multiplicity.

Theorem 11.1.8. Let f be analytic, non-constant on a region G, $z_0 \in G$, $w_0 = f(z_0)$, n the order of z_0 as a zero of $f(z) - w_0$. Then $\exists\, \epsilon > 0, \delta > 0$ such that $\forall\, c \in D(w_0, \delta) \backslash \{w_0\}$, the equation $f(z) - c = 0$ has n distinct roots in $D(z_0, \epsilon)$. In particular, $D(w_0, \delta) \subset f(G)$.

Proof. Choose ϵ_0 such that

1. $\overline{D(z_0, \epsilon_0)} \subset G$,
2. $f(z) - w_0 \neq 0$, $\forall\, z \in \overline{D(z_0, \epsilon_0)} \backslash \{z_0\}$

3. $f'(z) \neq 0$ for $z \in \overline{D(z_0, \epsilon_0)} \backslash \{z_0\}$.

Let $\epsilon < \epsilon_0$, and let $\gamma(t) = z_0 + \epsilon e^{it}$, $0 \leq t \leq 2\pi$. Let $\Gamma = f \circ \gamma$. By (2), $w_0 \notin \Gamma$. Choose $\delta > 0$ such that $D(w_0, \delta) \cap \Gamma = \emptyset$. Consider $c \in D(w_0, \delta) \backslash \{w_0\}$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{dw}{w - c} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t)) - c} dt = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - c} dz$$

i.e., the number of zeros of $f(z) - c$ inside γ, counted according to multiplicity. By (3), all of the zeros inside γ are simple. Then the integral $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-c} dz$ counts the number of distinct zeros inside γ. The winding number $n(\Gamma, c) = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w-c}$ is constant on components of $\mathbb{C} \backslash \Gamma$. By the above calculation, for $c = w_0$ we have $\frac{1}{2\pi i} \int_\Gamma \frac{dw}{w-w_0} = n$. Hence for $c \in D(w_0, \delta) \backslash \{w_0\}$, there are n distinct points in $D(z_0, \epsilon)$ such that $f(z) - c = 0$. In particular, $D(z_0, \epsilon) \subset f(G)$. ☐

Corollary 11.1.9 (Open Mapping Theorem)**.** If f is a non-constant analytic function, then f is an open mapping.

Theorem 11.1.10. Suppose f is analytic and injective on a region G. Then

1. $f'(z) \neq 0$ on G, and
2. The inverse mapping $f^{-1} = g : f(G) \to G$ is analytic and $g'(z) = \dfrac{1}{f'(g(z))}$.

Proof.

1. Suppose $f'(a) = 0$ for some $a \in G$. Then $f(z) - f(a)$ has a zero of order ≥ 2 at $z = a$, which implies by Theorem 6.8 that f is not injective. Thus $f'(a) \neq 0$ for $a \in G$.
2. By the Open Mapping Theorem, g is continuous. Consider any $z \neq a \in f(G)$. Then $1 = \frac{f(g(z))-f(g(a))}{z-a} = \frac{f(g(z))-f(g(a))}{g(z)-g(a)} \frac{g(z)-g(a)}{z-a}$ for $z \neq a$. But $\frac{f(g(z))-f(g(a))}{g(z)-g(a)} \to f'(g(a)) \neq 0$ as $z \to a$. Hence $\frac{g(z)-g(a)}{z-a} \to \frac{1}{f'(g(a))}$ as $z \to a$. ☐

Remark. $f' \neq 0$ on G does not imply that f is injective on G.

Theorem 11.1.11. Let $f_1, f_2, \cdots$ be analytic on G. If $f_n \to f$ uniformly on all compact subsets of G, then f is analytic and $f_n' \to f'$ uniformly on all compact subsets of G.

Proof. A uniform limit of continuous functions is continuous, so f is continuous. Suppose $\overline{D(a, r)} \subset G$. Let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. For each $n \geq 1$, $f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{\zeta - z} d\zeta$ for

$z \in D(a,r)$. For fixed z, $f_n(z) \to f(z)$. Given $\epsilon > 0, \exists\, N$ such that $|f - f_n| < \epsilon$ on γ for $n > N$. So,

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{\zeta - z} d\zeta \right| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) - f_n(\zeta)}{\zeta - z} d\zeta \right| \\
&\leq \frac{\epsilon}{\operatorname{dist}(z,\gamma)} \frac{1}{2\pi} 2\pi r \\
&= \frac{\epsilon r}{\operatorname{dist}(z,\gamma)}
\end{aligned}$$

and $\frac{\epsilon r}{\operatorname{dist}(z,\gamma)} \to 0$ as $\epsilon \to 0$. Passing to the limit, we get $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$. By 5.20, f is analytic on $D(a,r)$, and hence on G. For the second assertion it is sufficient to prove $f_n' \to f'$ on $\overline{D(a,r/2)}$ whenever $\overline{D(a,r/2)} \subset G$. Recall: $f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta$, $f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$. Given $\epsilon > 0, \exists\, N$ such that $|f - f_n| < \epsilon$ on γ. For $n \geq N$ and $z \in \overline{D(a,r/2)}$,

$$\begin{aligned}
|f'(z) - f_n'(z)| &= \frac{1}{2\pi} \left| \int_\gamma \frac{f(\zeta) - f_n(\zeta)}{(\zeta - z)^2} d\zeta \right| \\
&\leq \frac{1}{2\pi} \frac{\epsilon}{(r/2)^2} 2\pi r \\
&= \frac{4\epsilon}{r}
\end{aligned}$$

and $\frac{4\epsilon}{r} \to 0$ as $\epsilon \to 0$. Thus $f_n' \to f'$ uniformly on $\overline{D(a,r/2)}$. □

Theorem 11.1.12 (A Mean Value Theorem)**.** If f is analytic on G and $\overline{D(a,r)} \subset G$, then $f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$.

Proof. Let $\gamma(t) = a + re^{i\theta}$, $0 \leq \theta \leq 2\pi$. By Cauchy's Formula, $f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{(a + re^{i\theta}) - a} rie^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$. □

Theorem 11.1.13 (Maximum-Modulus Principle)**.** Let f be analytic on a region G. If $|f|$ attains a maximum value at some $a \in G$, then f is a constant.

Proof. Suppose $|f(z)| \leq |f(a)|$ for $z \in D(a,R)$. Consider any $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, for $0 < r < R$. Then

$$\begin{aligned}
|f(a)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(a + re^{i\theta}) \right| d\theta \\
&\leq \frac{1}{2\pi} |f(a)| \int_0^{2\pi} d\theta \\
&\leq |f(a)|
\end{aligned}$$

So $0 = \int_0^{2\pi} \left[|f(a)| - \left|f(a+re^{i\theta})\right|\right] d\theta$, from which we conclude $|f(a)| = |f(a+re^{it})|$, $0 \leq t \leq 2\pi$, i.e., $|f(z)|$ is constant on $D(a,r) \Rightarrow f$ is constant on $D(a,R)$ (becuase an analytic function with constant norm must be constant). ☐

Theorem (Maximum-Modulus Principle for Bounded Regions)**.** Let G be a bounded region. Assume f is analytic on G and continuous on $\overline{G}$. Then $|f|$ attains its max on ∂G. Alternately, if $|f| \leq M$ on ∂G, then $|f| \leq M$ on G.

Proof. Since f is continous on the compact set $\overline{G}$, $|f|$ attains a maximum value on $\overline{G}$. If it attains its maximum value on ∂G, we're done. If it attains its maximum value at $a \in G$, then f is constant, in which case $|f|$ is constant, and $|f|$ attains this maximum value on ∂G. ☐

Theorem (Maximum-Modulus Principle for Arbitrary Regions)**.** Let G be a region, and let Γ be its boundary in $\mathbb{C}_\infty$. Suppose f is analytic on G. If $\exists\, M \geq 0$ such that $\limsup_{z\to a} |f(z)| \leq M$ for all $a \in \Gamma$, then $|f| \leq M$ on G.

Theorem 11.1.14 (Schwarz's Lemma)**.** Let f be analytic, with $|f(z)| \leq 1$ on $D = D(0,1)$. If $f(0) = 0$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ on D. If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $f = cz$ on D for some $|c| = 1$.

Proof. Set

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Then g is analytic in D. Fix $z \in D$. For $|\zeta| = r < 1$, $|g(\zeta)| = \left|\frac{f(\zeta)}{\zeta}\right| \leq \frac{1}{r}$. By the Maximum-Modulus Principle, $|g(z)| \leq \frac{1}{r}$ for $|z| < r$. In particular, $|g(z)| \leq 1$ on D. Thus $\left|\frac{f(z)}{z}\right| \leq 1 \Rightarrow |f(z)| \leq |z|$ for $z \neq 0$. For $z = 0$, $|f'(0)| = |g(0)| \leq 1$. If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then g is constant by the Maximum-Modulus Principle. In this case, $f(z) = cz$ for some $|c| = 1$. ☐

Remark. An automorphism of the disk D is an analytic bijection $f : D \to D$. By 6.10, f^{-1} is also analytic and hence an autormorphism.

Theorem 11.1.15. The set of all automorphisms of D coincides with the set of functions $f(z) = c\frac{z-a}{1-\bar{a}z}$, with $|c| = 1$ and $|a| < 1$.

Proof. Suppose f has the above form. Without loss of generality, take $c = 1$. Set $w = \varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Then $z = \varphi_{-a}(w)$, so f is injective. On $|z| = 1$, $\left|\frac{z-a}{1-\bar{a}z}\right|^2 = 1$. Then by the Maximum-Modulus Principle, $\left|\frac{z-a}{1-\bar{a}z}\right| < 1$ on D. Both φ_a and $\varphi_a^{-1} = \varphi_{-a}$ thus map D into

D. Hence both are surjective, so f is an automorphism.

Conversely, suppose $f : D \to D$ is an automorphism. Let $a \in D$ be the unique point such that $f(a) = 0$. Then $h = f \circ \varphi_{-a}$ maps D onto D, and $h(0) = 0$. By Schwarz's Lemma, $|h(z)| \le |z|$ on D. That is, $|f(\varphi_{-a}(z))| \le |z|$ on D. Replace z by $\varphi_a(z)$. Then $|f(z)| \le |\varphi_a(z)| = \left|\frac{z-a}{1-\overline{a}z}\right|$. Also, $k = \varphi_a \circ f^{-1}$ is analytic by 6.10 and $k : D \to D$ satisfies $k(0) = 0$. By Schwarz's Lemma, $|k(z)| \le |z|$ on $D \Rightarrow |\varphi_a(f^{-1}(z))| \le |z|$ on D, and $\left|\frac{z-a}{1-\overline{a}z}\right| = |\varphi_a(z)| \le |f(z)|$ on D. Thus $|f(z)| = \left|\frac{z-a}{1-\overline{a}z}\right|$. Equivalently,

$$\left| f(z) \left(\frac{z-a}{1-\overline{a}z} \right)^{-1} \right| = 1$$

on $D \backslash \{a\}$. So $f(z) = c\frac{z-a}{1-\overline{a}z}$ for some $|c| = 1$. □

Theorem 11.1.16 (Other Forms of Schwarz's Lemma)**.** Let $f : D \to D$ be analytic. Then for $w, z \in D$, we have:

1. $\left| \dfrac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} \right| \le \left| \dfrac{z-w}{1-\overline{w}z} \right|$
2. $\left| \dfrac{f(z) - f(w)}{z-w} \right|^2 \le \dfrac{1 - |f(z)|^2}{1-|z|^2} \dfrac{1-|f(w)|^2}{1-|w|^2}$
3. $|f'(w)| \le \dfrac{1-|f(w)|^2}{1-|w|^2}$

Proof.

1. Set $\zeta = \frac{z-w}{1-\overline{w}z}$. Then $z = \frac{\zeta+w}{1+\overline{w}\zeta}$, and

   $$g(\zeta) = \frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} = \frac{f(\frac{\zeta+w}{1+\overline{w}\zeta}) - f(w)}{1 - \overline{f(w)}f(\frac{\zeta+w}{1+\overline{w}\zeta})}$$

   Now $g : D \to D$ is analytic and $g(0) = 0$. By Schwarz's Lemma, $|g(\zeta)| \le |\zeta|$.

2. By (1), $|f(z) - f(w)|^2 |1 - \overline{w}z|^2 \le |1 - \overline{f(w)}f(z)|^2 |z-w|^2$. The result then follows from the fact that $|1 - \overline{u}v|^2 = (1-|u|^2)(1-|v|^2) + |u-v|^2$.

3. Let $z \to w$ in (2). □

11.2 Sequences of Analytic Functions

Theorem 11.2.1 (Hurwitz's Theorem)**.** Let $f_1, f_2, \cdots$ be analytic on a region G, and suppose $f_n \to f$ uniformly on compact subsets of G. Suppose $\overline{D(a,r)} \subset G$ and $f(z) \neq 0$ for

$|z-a| = r$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, f_n and f have the same number of zeros in $D(a,r)$ counted according to multiplicity.

Proof. Let $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$. Since $f \neq 0$ on γ, $\exists\, \delta > 0$ such that $|f| > \delta$ on γ. Since $f_n \to f$ uniformly on γ, without loss of generality we may assume that $|f_n| > \delta/2$ on γ. Then

$$\begin{aligned} \left|\frac{f'}{f} - \frac{f_n'}{f_n}\right| &= \left|\frac{f' f_n - f_n' f_n + f_n' f_n + f f_n'}{f f_n}\right| \\ &\le \frac{|f_n||f' - f_n'| + |f_n'||f_n - f|}{\delta(\delta/2)} \end{aligned}$$

By 6.11, $f_n' \to f_n$ uniformly on γ. Hence $\frac{f_n'}{f_n} \to \frac{f'}{f}$ uniformly on γ, and we have

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

By 6.7, these integrals count the number of zeros according to multiplicity of f_n and f, respectively, in $D(a,r)$. So the sequence on the left is a convergent sequence of integers, which implies that it is eventually a constant sequence, and the result follows. ☐

Definition. Call an analytic function $f : G \to \mathbb{C}$ univalent if it is injective.

Remark. If $f : G \to \mathbb{C}$ is univalent, then $f'(z) \neq 0, \forall\, z \in G$.

Theorem 11.2.2. Let $f_1, f_2, \cdots$ be univalent on a region G, and assume $f_n \to f$ uniformly on compact subsets of G. Then either f is constant or f is univalent.

Proof. Assume that f is non-constant. Suppose f is not univalent. Then $\exists\, z_1, z_2 \in G$, $z_1 \neq z_2$, such that $f(z_1) = f(z_2)$. Set $g(z) = f(z) - f(z_1)$. Then $g(z_2) = 0$. Choose $\overline{D(z_2,r)} \subset G$ such that $z_1 \notin \overline{D(z_2,r)}$ and $g \neq 0$ on $|z - z_2| = r$. Note: g has $k \ge 1$ zeros in $D(z_2,r)$. Let $g_n(z) = f_n(z) - f_n(z_1), \forall\, n \in \mathbb{N}$. Then $g_n \to g$ uniformly on compact sets in G. By Hurwitz's Theorem, $\exists\, N > 0$ such that $\forall\, n > N$, g_n has k zeros in $D(z_2,r)$. For such n, f_n is not univalent ($\Rightarrow\Leftarrow$). Conclude that f must be univalent. ☐

Remark. Every analytic function f on a simply-connected region G has a primitive (i.e., an analytic function g such that $g' = f$). Suppose f is an analytic function on a simply connected region G. Every closed curve γ in G can be deformed to a point. By Cauchy's Theorem we get $\int_\gamma f = 0$ for every closed curve γ in G. Then by 5.10, f has a primitive.

Definition 11.2.3. Let f be analytic on a region G. By a branch of $\log f(z)$ we mean an analytic function g on G such that $e^{g(z)} = f(z)$. If $f(z) = z$, then by a branch of $\log f(z)$ we simply mean a branch of the logarithm.

Theorem. Let f be analytic and non-vanishing on a simply-connected region G. Then there exists an analytic branch of $\log f(z)$ on G.

Proof. Since $f \neq 0$ on G, $\frac{f'}{f}$ is analytic on G. Since G is simply connected, there exists an analytic function g on G with $g' = \frac{f'}{f}$. Then $(fe^{-g})' = f'e^{-g} - fg'e^{-g} = f'e^{-g} - f'e^{-g} = 0$. So fe^{-g} is constant. Write $f = ce^g = e^{b+g}$, where $c = e^b$. Then $g(z) + b$ is a branch of $\log f(z)$ on G. $\square$

Chapter 12

Singularities of Analytic Functions

12.1 Laurent Series

Definition 12.1.1 (Isolated Singularities)**.** Call $a \in \mathbb{C}$ an isolated singularity of an analytic function f if f is defined and analytic on $D(a, \delta)\backslash\{a\}$ for some $\delta > 0$, but is neither defined nor analytic at a. Call a a removable singularity if it is possible to define $f(a)$ so that f is analytic on $D(a, \delta)$. Call a a pole if $\exists\, N \in \mathbb{N}$ such that $f(z) = \frac{g(z)}{(z-a)^N}$ where $g(z)$ is analytic at a and $g(a) \neq 0$. In this case, we say that a is a pole of f of order N. Call a an essential singularity of f in all other cases.

Definition 12.1.2 (Singularities and Zeros at ∞)**.** Let f be analytic on $\{z : |z| > R\}$, for some $R > 0$. Call ∞ an isolated singularity of f if 0 is an isolated singularity of $g(z) = f(1/z)$. We classify ∞ as a singularity of f by classifying 0 as a singularity of g. Call ∞ a zero of f if $g(z)$ has a removable singularity at 0, and can be made analytic through the definition $g(0) = 0$. Call ∞ a zero of multiplicity N of f if $g(z)$ has a zero of multiplicity N at 0.

Theorem 12.1.3 (Riemann's Principle)**.** Suppose the analytic function f has an isolated singularity at a. If $\lim_{z\to a}(z-a)f(z) = 0$, then the singularity is removable.

Proof. Set

$$g(z) = \begin{cases} (z-a)f(z) & z \neq a \\ 0 & z = a \end{cases}$$

Then g is analytic in $D(a, \delta)\backslash\{a\}$ and continuous at a by the assumption. Then g is analytic on $D(a, \delta)$ by 5.24. Write $g(z) = \sum_{j=1}^{\infty} c_j (z-a)^j$. Then $f(z) = \frac{g(z)}{z-a} = \sum_{j=1}^{\infty} c_j (z-a)^{j-1}$. Define $f(a) = c_1$. Then f is analytic at a. $\square$

Definition 12.1.4 (Laurent Series)**.** Given two analytic functions $f_1(z) = \sum_{n=1}^{\infty} c_{-n} \frac{1}{(z-a)^n}$ and $f_2(z) = \sum_{n=1}^{\infty} c_n (z-a)^n$, with f_1 absolutely convergent on $\{z : |z-a| > R_1\}$ and

f_2 absolutely convergent on $\{z : |z-a| < R_2\}$, we define the analytic function f with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ on the annulus $0 \leq R_1 < |z-a| < R_2 \leq \infty$ by $f(z) = f_1(z) + f_2(z)$. We call $f_1(z)$ the principle part of f.

Theorem. Let f be analytic on $G = \{z : 0 \leq R_1 < |z-a| < R_2 \leq \infty\}$. Then $\forall\, n \in \mathbb{Z}$, $\exists\, c_n \in \mathbb{C}$ such that $f(z) = \sum_{-\infty}^{\infty} c_n(z-a)^n$. Moreover, the c_n are unique and are given by the formula $c_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta$, where $\gamma_r(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, and $R_1 < r < R_2$.

Proof. Take $a = 0$. Choose r_1, r_2 such that $R_1 < r_1 < r_2 < R_2$. Fix $z \in G$ with $r_1 < |z| < r_2$. Set

$$h(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

Then h is analytic on G. Let $\gamma_1(t) = r_1 e^{it}$, $\gamma_2(t) = r_2 e^{it}$, $0 \leq t \leq 2\pi$. By the deformation principle, $\int_{\gamma_1} h(\zeta)d\zeta = \int_{\gamma_2} h(\zeta)d\zeta$. Let $\int_{\gamma_{r_2}-\gamma_{r_1}} = \int_{\gamma_{r_1}} - \int_{\gamma_{r_2}}$. Then $\int_{\gamma_{r_2}-\gamma_{r_1}} h(\zeta)d\zeta = 0$. So

$$\int_{\gamma_{r_2}-\gamma_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\gamma_{r_2}-\gamma_{r_1}} \frac{f(z)}{\zeta - z} d\zeta = f(z) \cdot 2\pi i(1-0) = 2\pi i \cdot f(z)$$

Then $f(z) = f_1(z) + f_2(z)$, where $f_1(z) = \frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta$ and $f_2(z) = -\frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta$.On γ_{r_2}, $|\zeta| > |z|$, so $\frac{1}{\zeta - z} = \frac{1}{\zeta}\frac{1}{1-z/\zeta} = \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}}$. Then

$$\begin{aligned} f_2(z) &= \frac{1}{2\pi i}\int_{\gamma_{r_2}} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}} f(\zeta) d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \right) z^n \\ &= \sum_{n=0}^{\infty} c_n z^n \end{aligned}$$

where $c_n = \frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$, and interchanging the order of integration and summation is justified by uniform convergence. On γ_{r_1}, $|\zeta| < |z|$, so $-\frac{1}{\zeta - z} = \frac{1}{z - \zeta} = \frac{1}{z}\frac{1}{1-\zeta/z} = \sum_{k=0}^{\infty} \frac{\zeta^k}{z^{k+1}}$, and

$$\begin{aligned} f_1(z) &= \frac{1}{2\pi i}\int_{\gamma_{r_1}}\sum_{k=0}^{\infty}\frac{\zeta^k}{z^{k+1}}f(\zeta)d\zeta \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_{r_1}}\zeta^k f(\zeta)d\zeta\right)z^{-(k+1)} \\ &= \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_{r_1}}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right)z^n \\ &= \sum_{n=-\infty}^{-1}c_n z^n \end{aligned}$$

where $c_n = \frac{1}{2\pi i}\int_{\gamma_{r_1}}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta$, and interchanging the order of summation and integration is justified by uniform convergence. The result then follows. □

Remark 12.1.5 (Classification of Singularities)**.** Let f have an isolated singularity at a. Then in some set $D(a,\delta)\backslash\{a\}$, we have $f(z)=\sum_{-\infty}^{\infty}c_n(z-a)^n$.

1. If $c_n = 0\,\forall\, n<0$, then a is a removable singularity of f.
2. If $c_n = 0\,\forall\, n<-N$, $c_{-N}\neq 0$ for some $N\in\mathbb{N}$, then a is a pole of order N of f.
3. If $c_{-n}\neq 0$ for infinitely many $n\in\mathbb{N}$, then a is an essential singularity of f.

Suppose ∞ is an isolated singularity of f. Set $g(z)=f(1/z)$. Write $g(z)=\sum_{-\infty}^{\infty}d_n z^n$ for $0<|z|<\delta$ and some $\delta>0$. Then $f(z)=\sum_{-\infty}^{\infty}c_n z^n$ for $|z|>\frac{1}{\delta}$, where $c_n = d_{-n}$.

1. If $c_n = 0\,\forall\, n<0$, ∞ is a removable singularity of f.
2. If $c_n = 0\,\forall\, n<N$, $c_N\neq 0$ for some $N\in\mathbb{N}$, then ∞ is a pole of order N of f.
3. If $c_n\neq 0$ for infinitely many $n\in\mathbb{N}$, then ∞ is an essential singularity of f.

Remark. Note that if f has a pole at a, then $|f(z)|\to\infty$ as $z\to a$. If f has a removable singularity at a, then for some $M>0$, $|f(z)|\leq M$ in a neighborhood of a. If a is an essential singularity of f, the behavior of f near a is more complicated.

Theorem 12.1.6 (Casaroti-Weierstrass Theorem)**.** Suppose f has an essential singularity at a. then for any $\Delta = D(a,r)\backslash\{a\}$, $f(\Delta)$ is dense in $\mathbb{C}$.

Proof. Let f be analytic on $\Delta = D(a,\delta)$ except for an essential singularity at a. Suppose $f(\Delta)$ is not dense in $\mathbb{C}$. Then $\mathbb{C}\backslash f(\Delta)$ contains a disk $D(b,r)$ for some $b\in\mathbb{C}$ and $r>0$.

Then for every $z \in \Delta$, $|z-b| > r$ and $\left|\frac{1}{f(z)-b}\right| \leq \frac{1}{r}$. Let $g(z) = \frac{1}{f(z)-b}$. Evidently the function g is analytic and bounded on Δ, and has an isolated singularity at $z = a$. By Riemann's Principle, this singularity is removable. Define $g(a)$ so that g is analytic at a. Then $f(z) = b + \frac{1}{g(z)}$. Now either $g(a) \neq 0$, in which case f has a removable singularity at a ($\Rightarrow\Leftarrow$), or $g(a) = 0$, in which case f has a pole at a ($\Rightarrow\Leftarrow$). Conclude that the original assumption was false, and that $f(\Delta)$ is dense in $\mathbb{C}$. $\square$

Definition 12.1.7. A function f is said to be meromorphic on the region G if it is analytic on G except for poles.

Theorem 12.1.8 (Partial Fraction Decomposition)**.** Let f be meromorphic on $\mathbb{C}_\infty$. Then f has only a finite number of poles in $\mathbb{C}_\infty$, say $a_1, \cdots, a_r \in \mathbb{C}_\infty$. If $f_1(z), \cdots, f_r(z)$ are the principle parts of f at the poles $a_1, \cdots, a_r$, then $f(z) = f_1(z) + \cdots + f_r(z) + c$ for some constant $c \in \mathbb{C}$. In particular, f is a rational function (i.e., $f(z) = \frac{p(z)}{q(z)}$ for some polynomials $p(z)$ and $q(z)$, $q(z) \neq 0$).

Proof. Let f be meromorphic on $\mathbb{C}_\infty$. Suppose f has infinitely many poles in $\mathbb{C}_\infty$. Then the poles of f must have a limit point in $\mathbb{C}_\infty$, say $b \in \mathbb{C}_\infty$. Then b is not a pole of f, because f is not analytic in any deleted neighborhood of b, nor is f analytic at b. Indeed, we can choose a sequence of points $z_k \to b$ such that f is analytic at each z_k and $|f(z_k)| \to \infty$. If f were analytic at b, then it would be continuous at b, in which case we'd have $\infty = \lim_{k\to\infty} |f(z_k)| = |f(b)| < \infty$, a contradiction. Conclude that f can have only finitely many poles in $\mathbb{C}_\infty$. Let $a_1, \cdots, a_r$ be the poles of f, and let $f_1(z), \cdots, f_r(z)$ be the corresponding principle parts of f. Then $g(z) = f(z) - f_1(z) - \cdots - f_r(z)$ has only removable singularities in $\mathbb{C}_\infty \Rightarrow g$ is a bounded entire function $\Rightarrow g$ is constant by Liouville's Theorem. The result then follows. $\square$

Remark. Suppose f has an isolated singularity at $a \in \mathbb{C}$. Let γ be a curve winding once around a. Suppose $f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n$ for $0 < |z-a| < \delta$. Then $\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{-\infty}^{\infty} c_n \frac{1}{2\pi i}\int_\gamma (z-a)^n dz$. For $n \neq -1$, $(z-a)^n$ has a primitive $\frac{1}{n+1}(z-a)^{n+1}$. Thus $\frac{1}{2\pi i}\int_\gamma f(z)dz = c_{-1}\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz = c_{-1}$.

12.2 Residue Theorem and its Consequences

Definition 12.2.1 (Residue of an Analytic Function)**.** Let $f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n$ be analytic with an isolated singularity at $z = a$. We call the coefficient c_{-1} the residue of f at a, and denote it by $\operatorname{Res}(f; a)$.

Theorem 12.2.2 (Residue Theorem)**.** Let f be analytic on a simply connected region G except for isolated singularities at $a_1, \cdots, a_r \in \mathbb{C}$. Let γ be a closed curve in G not passing through any singularities of f. Then $\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{r} n(\gamma, a_k) \cdot \operatorname{Res}(f; a_k)$.

Proof. For each $j = 1, \cdots, r$, write $f(z) = f_j^-(z) + f_j^+(z)$, where $f_j^-(z) = \sum_{n=-\infty}^{-1} c_{j,n}(z-a_j)^n$ and $f_j^+(z) = \sum_{n=0}^{\infty} c_{j,n}(z-a_j)^n$. Set $h(z) = f(z) - f_1^-(z) - \cdots - f_r^-(z)$. Then h is analytic on G except for removable singularities at $a_1, \cdots, a_r$. By Cauchy's Theorem, $\int_\gamma h(z)dz = 0$. So

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma f(z)dz &= \frac{1}{2\pi i}\sum_{j=1}^{r}\int_\gamma f_j^-(z)dz \\ &= \frac{1}{2\pi i}\sum_{j=1}^{r}\int_\gamma c_{j,-1}\frac{1}{z-a_j}dz \\ &= \frac{1}{2\pi i}\sum_{j=1}^{r} c_{j,-1} \cdot n(\gamma, a_j) \\ &= \frac{1}{2\pi i}\sum_{j=1}^{r} n(\gamma, a_j) \cdot \operatorname{Res}(f; a_j) \end{aligned}$$

□

Remark 12.2.3 (Calculation of Residues)**.** Let f have an isolated singularity at a.

1. If a is a simple pole of f, then $\operatorname{Res}(f; a) = \lim_{z\to a}(z-a)f(z)$. If $f(z) = \frac{g(z)}{h(z)}$, where g and h are analytic at a, $g(a) \neq 0$, and h has a simple zero at a, then $\operatorname{Res}(f; a) = \frac{g(a)}{h'(a)}$.
2. If f has a pole of order n at a, then $\operatorname{Res}(f; a) = \frac{1}{(n-1)!} \cdot \frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)\Big|_{z=a}$.
3. If f has an essential singularity at a,the easiest way to compute $\operatorname{Res}(f; a)$ is often to determine the Laurent series of f at a.

Example 12.2.4 (Evaluation of Definite Integrals)**.** Evaluate certain definite Riemann integrals by considering an appropriate complex-valued function, and then integrating that function along a countour intersecting the real axis in the intervals over which the definite Riemann integral is evaluated.

1. $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$, where P and Q are polynomials and $\deg Q \geq 2 + \deg P$.
2. $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\left\{\begin{smallmatrix}\sin ax\\ \cos ax\end{smallmatrix}\right\}dx$.
3. $\int_{-\pi}^{\pi} R(\cos\theta, \sin\theta)\,d\theta = \int_\gamma R\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz}$, where $R(x, y)$ is a rational function of x and y, and $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$.

4. $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. Note that this integral exists as an improper Riemann integral, but not as a Lebesgue integral.

Theorem 12.2.5 (Evaluation of Sums)**.** Let f be analytic in $\mathbb{C}$ except for a finite number of poles $z_1, \cdots, z_r$, and $|f(z)| \leq \frac{c}{|z|^2}$ when $|z| > R$ for some $c > 0$ and $R > 0$. Then

1. $\sum_{\substack{n \in \mathbb{Z} \\ n \neq z_1, \cdots, z_r}} f(n) = -\sum_{k=1}^r \operatorname{Res}\left(f(z)\pi\cot(\pi z); z_k\right)$
2. $\sum_{\substack{n \in \mathbb{Z} \\ n \neq z_1, \cdots, z_r}} (-1)^n f(n) = \sum_{k=1}^r \operatorname{Res}\left(f(z)\pi\csc(\pi z); z_k\right)$

Remark. Note that $\pi\cot(\pi z)$ has simple poles at each integer n, and $\operatorname{Res}(\pi\cot(\pi z); n) = 1$. Similarly, $\pi\csc(\pi z)$ has simple poles at each integer n, and $\operatorname{Res}(\pi\csc(\pi z); n) = (-1)^n$.

Proof. Let γ_N be the polygonal path $[N+1/2+i(N+1/2), -(N+1/2)+i(N+1/2), -(N+1/2)-i(N+1/2), N+1/2-i(N+1/2)]$, with N large enough that γ_N encloses all of $z_1, \cdots, z_r$. Claim: $|\cot(\pi z)| \leq 2$ on γ. On the right edge, $z = N + 1/2 + iy$. Then

$$\begin{aligned}
|\cot(\pi z)| &= \left| i \frac{e^{\pi i - 2\pi y} + 1}{e^{\pi i - 2\pi y} - 1} \right| \\
&= \left| \frac{-e^{-2\pi y} + 1}{-e^{-2\pi y} - 1} \right| \\
&= \frac{|e^{-2\pi y} - 1|}{e^{-2\pi y} + 1} \\
&\leq 1
\end{aligned}$$

Since $\cot(\pi z)$ is an odd function, we have $|\cot(\pi z)| \leq 1$ on the left side as well. On the upper edge, $z = x + i(N + 1/2)$. Then

$$\begin{aligned}
|\cot(\pi z)| &= \left| i \frac{e^{2\pi i(x+i(N+1/2))} + 1}{e^{2\pi i(x+i(N+1/2))} - 1} \right| \\
&\leq \frac{e^{-2\pi(N+1/2)} + 1}{1 - e^{-2\pi(N+1/2)}} \\
&= \frac{2}{1 - e^{-2\pi(N+1/2)}} - 1 \\
&\leq \frac{2}{1 - e^{-\pi}} - 1 \\
&< 2
\end{aligned}$$

And similarly for the bottom edge. Then by the Residue Theorem,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_N} f(z)\pi\cot(\pi z)\,dz &= \sum\{\text{residues of poles inside } \gamma_N\}\\ &= \sum_{\substack{n=-N\\ n\neq z_1,\cdots,z_r}}^{N} f(n)\cdot 1+\sum_{k=1}^{r}\operatorname{Res}\left(f(z)\pi\cot(\pi z);z_k\right)\end{aligned}$$

But for $N > R$, $|f(z)\pi\cot(\pi z)| \leq \frac{c}{N^2}\cdot 2$ on γ_N. Noting that $L(\gamma_N) = 8(N+1/2)$, we have $\frac{1}{2\pi i}\int_{\gamma_N} f(z)\pi\cot(\pi z)\,dz \to 0$ as $N\to\infty$ by the ML-formula, and the first formula follows. For the second formula, we argue similarly as in the first case to show that $\csc(\pi z)$ is bounded by $\sqrt{5}$ on γ_N. □

Example. $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

Remark. Recall that if $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$, and f is analytic on $D(a,r)$ with no zeros on γ, then $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz$ is equal to the number of zeros of f inside γ, counted according to multiplicity.

Theorem 12.2.6 (Argument Principle)**.** Let f be meromorphic on a simply connected region G with distinct zeros $a_1, a_2, \cdots$ and distinct poles $b_1, b_2, \cdots$ in G. Let γ be a closed curve in G not passing through any zeros or poles of f. Then $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = Z_f - P_f$, where $Z_f = \sum_j n(\gamma, a_j)\cdot\operatorname{ord}(a_j)$, and $P_f = \sum_k n(\gamma, b_k)\cdot\operatorname{ord}(b_k)$.

Proof. Apply the Residue Theorem to $\frac{f'}{f}$. Consider the zero a_j of multiplicity n_j. Then $f = (z-a_j)^{n_j}g(z)$ for some analytic function g on G with $g(a_j)\neq 0$. Then $f'(z) = n_j(z-a_j)^{n_j-1}g(z)+(z-a_j)^{n_j}g'(z) \Rightarrow \frac{f'}{f} = \frac{n_j}{z-a_j}+\frac{g'}{g}$ and $\frac{g'}{g}$ is analytc at a_j. So $\operatorname{Res}(f;a_j) = n_j$. Considering the pole b_j of order m_j, we similarly have $f(z) = (z-b_j)^{-m_j}h(z)$ for some analytic function h on G with $h(b_j)\neq 0$. Then $\frac{f'}{f} = \frac{-m_j}{z-b_j}+\frac{h'}{h}$, so $\operatorname{Res}(f;b_j) = -m_j$. The result follows. □

Remark. To say that f is analytic on an inside a simple closed curve γ, we mean that γ is the boundary of a region G, $n(\gamma,a) = 1, \forall\, a\in G$, and f is analytic on a region containing $\overline{G} = G\cup\gamma$.

Remark (Connection with $\arg f(z)$)**.** Observe that we can define $\arg f(z)$ continuously on γ except for one jump. That is, we can define $\alpha(t) = \arg f(\gamma(t))$ continuously for $a \leq t \leq b$. Cover γ by a finite number of disks such that no disk contains a zero or pole of f. Define an analytic branch of $\log f(z)$ on each disk Ω_j. So $\arg f(z) = \operatorname{Im}(\log f(z))$ is continuous on Ω_j. Claim: $\frac{1}{2\pi i}\int_\gamma\frac{f'}{f}\,dz = \frac{1}{2\pi}\Delta_\gamma \arg f(z)$, where $\Delta_\gamma\arg f(z) = \alpha(b)-\alpha(a)$, the change in $\arg f(z)$ around γ.

Proof. Consider a segment γ_{t_1,t_2} in Ω_j. Then $\int_{\gamma_{t_1,t_2}} \frac{f'}{f}\,dz = \log f(\gamma(t_2)) - \log f(\gamma(t_1))$. So

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz &= \frac{1}{2\pi i}\left(\log f(\gamma(b)) - \log f(\gamma(a))\right)\\ &= \frac{1}{2\pi i}\left(i\arg f(\gamma(b)) - i\arg f(\gamma(a))\right)\\ &= \frac{1}{2\pi}\Delta_\gamma \arg f(z)\end{aligned}$$

□

Theorem 12.2.7 (Rouche's Theorem)**.** Let f, g be meromorphic on and inside a simple closed curve γ, and assume that there are no poles of f or g on γ. Suppose $|f(z)+g(z)| < |f(z)|+|g(z)|$ on γ. Then $Z_f - P_f = Z_g - P_g$.

Proof. Note: f, g can have no zeros on γ, or else the inequality $|f(z)+g(z)| < |f(z)|+|g(z)|$ is false for some point $z \in \gamma$. Now on γ, $\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}\right|+1$. Furthermore, $\frac{f(z)}{g(z)}$ cannot be a non-negative real number for any $z \in \gamma$, or else the assumed inequality fails. Hence, the image of γ under $\frac{f}{g}$ lies in $\mathbb{C}\backslash[0,\infty)$. Let $\log z$ be an analytic branch of the logarithm on $\mathbb{C}\backslash[0,\infty)$. Then $\log\left(\frac{f(z)}{g(z)}\right)$ is analytic in a neighborhood of γ, and in fact is a primitive for $\frac{f'}{f} - \frac{g'}{g}$ in such a neighborhood of γ. Then $\int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{g'}{g}\,dz$ since $\int_\gamma \frac{f'}{f} - \frac{g'}{g}\,dz = 0$, and the result follows from the argument principle. □

Example (Applications of the Argument Principle and Rouche's Theorem)**.**

1. The function $z^5 - 3z + 1$ has one root in $\{z : |z| < 1\}$ and 4 roots in $\{z : 1 < |z| < 2\}$. On $|z| = 1$, let $f(z) = z^5 - 3z + 1$, and let $g(z) = 3z - 1$. On $|z| = 2$, let $f(z) = z^5 - 3z + 1$, and let $g(z) = -z^5$.

2. How many roots of $z^6 - 5z^4 + 3z^2 - 1$ lie in $\{z : |z| < 1\}$? Let $f(z) = z^6 - 5z^4 + 3z^2 - 1$, and let $g(z) = 5z^4$. Note that $|c_1 + c_2| \le |c_1| + |c_2|$, with equality iff c_1 and c_2 lie on the same ray from the origin. So the inequality $|f(z) + g(z)| \le 5 = |5z^4| < |g(z)| + |f(z)|$ is strict on $|z| = 1$ except at possibly $z = \pm i$. For $z = \pm i$, $|f(z) + g(z)| = 5 < 10 = |f(z)| + |g(z)|$. Hence $z^6 - 5z^4 + 3z^2 - 1$ has four roots in $\{z : |z| < 1\}$.

3. Second proof of the fundamental theorem of algebra: Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, $n \ge 1$. Then $\frac{p(z)}{a_n z^n} \to 1$ as $z \to \infty$. Hence on $|z| = R$ for $R \gg 0$, $\left|\frac{p(z)}{a_n z^n} - 1\right| < 1$, or, $|p(z) - a_n z^n| < |a_n z^n|$. Then by Rouche's Theorem, for $R \gg 0$, p has n zeros in $\{z : |z| < R\}$ since $a_n z^n$ has n zeros on that set.

4. Apply the Argument Principle to show that $f(z) = z^4 + 2z^3 - 2z + 10$ has one root in each of the four quadrants by considering $\Delta_\gamma \arg f(z)$, where γ is the quarter circle of radius R in the first quadrant. Write $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, where $\gamma_1 = [0, R]$, γ_2 is the arc from R to iR, and $-\gamma_3 = [0, iR]$. Since f is real-valued on γ_1, $\Delta_{\gamma_1} \arg f(z) = 0$. On γ_2, $f(z) = z^4(1 + 2/z - 2/z^3 + 10/z^4)$ and $\Delta_{\gamma_2} \arg f(z) = 4 \cdot \frac{\pi}{2} + o(1)$. Note that $f(iR) = (R^4 + 10)\left(1 + \frac{-2i(R^3+R)}{R^4+10}\right)$, so $\arg f(iR) = o(1)$. On γ_3, $\operatorname{Re}(f(z)) \geq 10$. Since $f(0) = 10$, we conclude that $\Delta_{\gamma_3} \arg f(z) = o(1)$. Then $\Delta_\gamma \arg f(z) = 0 + 2\pi + o(1) + o(1) + 2\pi = 2\pi + o(1)$. But $\Delta_\gamma \arg f(z)$ must be an integer multiple of 2π, so we conclude $\Delta_\gamma \arg f(z) = 2\pi$ as $R \to \infty$. Thus for R large enough, f has exactly one zero inside γ. Argue similarly for the second quadrant, and argue for the third and fourth quadrants using symmetry.

Remark. Suppose f is analytic, not identically zero, and has a zero of order p at a. Then $\frac{f'}{f} = \frac{p}{z-a} + \sum_{k=0}^{\infty} c_k(z-a)^k$ for some constants c_k. If g is analytic at a, then $g\frac{f'}{f} = g\frac{p}{z-a} + g\sum_{k=0}^{\infty} c_k(z-a)^k$, and $\operatorname{Res}\left(g\frac{f'}{f}; a\right) = g(a)\cdot p$. If f has a pole of order q at b, and g is analytic at b, then $g\frac{f'}{f} = g\frac{-q}{z-a} + g(\sum_{k=0}^{\infty} d_k(z-a)^k)$ for some constants d_k, and $\operatorname{Res}\left(g\frac{f'}{f}; b\right) = g(b)\cdot(-q)$.

Theorem 12.2.8. Let f be meromorphic on and inside a simple closed curve γ, with no zeros or poles on γ, and let g be analytic on and inside γ. Then $\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{r} g(a_j) - \sum_{k=1}^{s} g(b_k)$, where $a_1, \ldots, a_r$ are the zeros of f inside γ repeated according to multiplicity, and $b_1, \ldots, b_s$ are the poles of f inside γ repeated according to multiplicity.

Example.

1. Let f be analytic on and inside a simple closed curve γ, with no zeros on γ. Then for $k = 0, 1, \cdots$, $\frac{1}{2\pi i}\int_\gamma z^k\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{r}(a_j)^k$, where $a_1, \ldots, a_r$ are the zeros of f inside γ, repeated according to multiplicity.

2. Let f be univalent on a region containing $\overline{D(a,r)}$, and let $\Omega = f(D(a,r))$. Let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Then for each $w \in \Omega$, $f^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \zeta\frac{f'(\zeta)}{f(\zeta)-w}\,d\zeta$. This follows from the previous theorem by considering the function $h(\zeta) = f(\zeta) - w$, which, since f is univalent, has exactly one root (namely $f^{-1}(w)$) in $\overline{D(a,r)}$.

Chapter 13

Compactness and the Riemann Mapping Theorem

Definition 13.0.9. Call two regions G_1 and G_2 conformally equivalent if there exists a univalent function f mapping G_1 onto G_2.

Example.

1. The domain and range of any linear fractional transformation are conformally equivalent.

2. The Koebe function $f(z) = \frac{z}{(1-z)^2}$ is univalent on the unit disc, and maps the unit disc to $\mathbb{C}\backslash(-\infty, -1/4]$. Thus the unit disk is conformally equivalent to $\mathbb{C}\backslash(-\infty, -1/4]$. Note that $f(z) = \sum_{k=1}^{\infty} kz^k$, and also $f(z) = \left(\frac{1}{2}\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$.

Theorem 13.0.10 (Vitali's Theorem)**.** Let $f_1, f_2, \ldots$ be analytic on a region G. Assume that $\exists\, M > 0$ such that for every compact subset $S \subset G$, $|f_n(z)| \leq M, \forall\, z \in S$ and $\forall\, n \in \mathbb{N}$. Let $z_1, z_2, \ldots$ be distinct points in G such that $z_k \to z_0$ for some $z_0 \in G$. If $\lim_{n\to\infty} f_n(z_j)$ exists for each $j \in \mathbb{N}$, then there exists an analytic function f on G such that $f_n \to f$ uniformly on all compact subsets of G.

Proof. First show that $f_1, f_2, \ldots$ is uniformly convergent in some disk $D(z_0, r_0)$. Take $z_0 = 0$ (to simplify the notation). Choose $R > 0$ such that $D(0, R) \subset G$. Let $0 < r_0 < r < R$. Choose $M > 0$ such that $|f_n(z)| \leq M$ for all $|z| \leq r$ and $n \geq 1$. Write $f_n(z) = a_{n_0} + a_{n_1}z + a_{n_2}z^2 + \cdots$. Then $|f_n(z) - f_n(0)| \leq 2M$. Apply Schwarz's Lemma to the function $h(z) = \frac{f_n(rz)-f_n(0)}{2M}$ to get that $|f_n(z) - f_n(0)| \leq \frac{2M}{r}|z|$ when $|z| \leq r$ and $n \geq 1$. Then if

$|z_j| \leq r$ we get

$$\begin{aligned} |a_{m_0} - a_{n_0}| &\leq |f_m(0) - f_m(z_j)| + |f_m(z_j) - f_n(z_j)| + |f_n(z_j) - f_n(0)| \\ &\leq \frac{4M}{r}|z_j| + |f_m(z_j) - f_n(z_j)| \end{aligned}$$

Now given $\epsilon > 0$, choose $j \in \mathbb{N}$ such that $\frac{4M}{r}|z_j| < \frac{\epsilon}{2}$. With j fixed, choose $N \in \mathbb{N}$ such that for $m, n > N$, $|f_m(z_j) - f_n(z_j)| < \frac{\epsilon}{2}$. (We can do this since the sequence $f_1(z_j), f_2(z_j), \dots$ is Cauchy for fixed j.) Then if $m, n > N$, we have $|a_{m_0} - a_{n_0}| < \epsilon$. Conclude that $\lim_{n\to\infty} a_{n_0} = a_0$ exists. Set $g_n(z) = \frac{f_n(z) - f_n(0)}{z}$. Then for all $|z| \leq r$ and $n \geq 1$, $|g_n(z)| \leq \frac{2M}{r}$. Clearly $\lim_{n\to\infty} g_n(z_j)$ exists for fixed $j \in \mathbb{N}$. Also, $g_n(z) = a_{n_1} + a_{n_2}z + a_{n_3}z^2 + \cdots$. Then by a similar argument as above, we have that $\lim_{n\to\infty} a_{n_1} = a_1$ exists, and by induction $\lim_{n\to\infty} a_{n_k} = a_k$ exists for each $k \in \mathbb{N}$. Set $f(z) = \sum_{k=0}^{\infty} a_k z^k$. By the Cauchy Estimates, $|a_{n_k}| \leq \frac{2M}{r^k}, \forall\, n \geq 1, k \geq 0$, hence $|a_k| \leq \frac{M}{r^k}, \forall\, k \geq 1$. So f converges for $|z| < r$. Now

$$\begin{aligned} |f - f_n| &= \left| \sum_{k=0}^{\infty} (a_k - a_{n_k}) z^k \right| \\ &\leq \sum_{k=0}^{P} |a_k - a_{n_k}| r_0^k + \sum_{k=P+1}^{\infty} \frac{2M}{r^k} r_0^k \end{aligned}$$

Given $\epsilon > 0$, first choose P such that $\sum_{k=P+1}^{\infty} \frac{2M}{r^k} r_0^k < \frac{\epsilon}{2}$. Then choose $N \in \mathbb{N}$ such that for $n > N$, $\sum_{k=0}^{P} |a_k - a_{n_k}| r_0^k < \frac{\epsilon}{2}$. Then for $n > N$, $|f - f_n| < \epsilon$, and $f_n \to f$ uniformly on $|z| \leq r_0$. To complete the argument, apply compactness arguments to join any two points in G using a finite number of disks. ☐

Definition 13.0.11. A family $\mathfrak{F}$ of analytic functions on a region G is called normal if every sequence $\{f_n\}_1^{\infty}$ in $\mathfrak{F}$ has a subsequence that converges uniformly on all compact subsets of G.

Theorem 13.0.12 (Montel's Theorem)**.** Let $\mathfrak{F}$ be a family of analytic functions on a region G. If $\exists\, M > 0$ such that $|f| \leq M$ on S for every $f \in \mathfrak{F}$ and every compact subset $S \subset G$, then $\mathfrak{F}$ is normal.

Proof. Pick distinct points z_j converging to $z_0 \in G$. Let $\{f_n\}$ be a sequence in $\mathfrak{F}$. By the diagonal process, choose a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}$ exists $\forall\, j$. Then by Vitali's Theorem, $\lim_{k\to\infty} f_{n_k}$ exists uniformly on all compact subsets of G. ☐

Theorem 13.0.13 (Riemann Mapping Theorem)**.** Let G be a simply connected region, $G \neq \mathbb{C}$, and let $a \in G$. Then there exists a unique univalent function mapping G onto $D = \{z : |z| \leq 1\}$ such that $f(a) = 0$ and $f'(a) > 0$.

Proof. Let $\mathfrak{F}$ be the family of all functions f on G such that (i) f is analytic, (ii) f is univalent, (iii) $f(G) \subset D$, and (iv) $f(a) = 0$ and $f'(a) > 0$. *Existence:*

1. $\mathfrak{F} \neq \emptyset$. Since $G \neq \mathbb{C}$, we can choose $b \notin G$. Then $z - b$ is analytic and non-zero on G. By 6.19, there exists a branch of $\log(z - b)$ on G. Let $g_0(z) = \log(z - b)$, and set $g(z) = e^{\frac{1}{2}g_0(z)}$. One checks that g is univalent on G.

   - Claim: $g(G)$ omits a disk $\Delta \subset \mathbb{C}$. Reasoning: Note that $g(a) \in g(G)$ and $g(G)$ is open. Choose $r > 0$ such that $D(g(a), r) \subset g(G)$. We show that $D(-g(a), r) \cap g(G) = \emptyset$. Suppose this is false. Then $\exists\, z_0 \in G$ such that $g(z_0) \in D(-g(a), r)$. Then $|g(a) - (-g(z_0))| = |-g(a) - g(z_0)| < r$, so $-g(z_0) \in D(g(a), r) \subset g(G)$. Hence $-g(z_0) = g(w_0)$ for some $w_0 \in G$. Then $(-g(z_0))^2 = (g(w_0))^2 \Rightarrow z_0 - b = w_0 - b \Rightarrow z_0 = w_0$. Then $-g(z_0) = g(z_0) \Rightarrow g(z_0) = 0$, a contradiction because g is non-vanishing (the exponential function is non-vanishing). We conclude that $g(G)$ omits a disk $\Delta \subset \mathbb{C}$.

   Choose a linear fractional transformation $Tz = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, such that $T(\mathbb{C}\backslash\Delta) \subset D$. (Such a T exists.) Let $h = T \circ g$. Then h is univalent and maps G into D. Let $f(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z)-h(a)}{1-\overline{h(a)}h(z)}$. Then $f(a) = 0$ and $f'(a) = \frac{|h'(a)|}{1-|h(a)|^2} > 0$. (Note that univalent functions have non-vanishing derivatives.) Then $f \in \mathfrak{F}$, and $\mathfrak{F} \neq \emptyset$.

2. Let $m = \sup_{f\in\mathfrak{F}} f'(a)$. Choose $r > 0$ such that $D(a, r) \subset G$. On $|z - a| = r$, $\left|\frac{f(z)-f(a)}{z-a}\right| = \frac{|f(z)|}{r} \leq \frac{1}{r}$ for any $f \in \mathfrak{F}$. By the Maximum Modulus Principle, the bound extends inside $D(a, r)$. Then taking the limit $z \to a$, we have $|f'(a)| \leq \frac{1}{r}$. So $m < \infty$. Choose $\{f_n\}$ in $\mathfrak{F}$ such that $f_n'(a) \to m$. By Montel's Theorem, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges uniformly on compact subsets of G to some function φ. By 6.11, $f_{n_k}' \to \varphi'$ uniformly on compact subsets of G, so $\varphi'(a) = m$. Noe that φ is non-constant since $\varphi'(a) > 0$, so by 6.18 φ is univalent. Also, $\varphi(a) = \lim_{k\to\infty} f_{n_k}(a) = 0$, so $\varphi \in \mathfrak{F}$.

3. $\varphi(G) = D$. Suppose that this is false. Choose $w \in D\backslash\varphi(G)$. Set $g(z) = \frac{\varphi(z)-w}{1-\overline{w}\varphi(z)}$. Then $g(G) \subset G$, and $g(z) \neq 0, \forall\, z \in G$. Choose an analytic branch of $\log g(z)$, and set $h(z) = e^{\frac{1}{2}\log g(z)}$. Note that $w \neq 0$ since $0 = \varphi(a) \in \varphi(G)$. Also, h is univalent. Set $f(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z)-h(a)}{1-\overline{h(a)}h(z)}$. Then $f \in \mathfrak{F}$ ($f(a) = 0$, f is univalent, and $f'(a) = \frac{|h'(a)|}{1-|h(a)|^2} > 0$). From $h(z)^2 = g(z) = \frac{\varphi(z)-w}{1-\overline{w}\varphi(z)}$ we get $2h(a)h'(a) = (1-|w|^2)\varphi'(a)$.

Also, $h(a)^2 = g(a) = -w$. Then

$$\begin{aligned} f'(a) &= \frac{|h'(a)|}{1-|h(a)|^2} \\ &= \frac{1}{1-|w|}\frac{(1-|w|^2)\varphi'(a)}{2(|w|)^{1/2}} \\ &= \frac{1+|w|}{2(|w|)^{1/2}}\varphi'(a) > \varphi'(a) \end{aligned}$$

since $\frac{1+|w|}{2(|w|)^{1/2}} > 1$. This is impossible, so the assumption was false, and we have that $\varphi(G) = D$.

Uniqueness: If φ, ψ are two such mappings, then $\varphi \circ \psi^{-1}$ is an automorphism of D, and $(\varphi \circ \psi^{-1})(0) = 0$. So $(\varphi \circ \psi^{-1})(z) = cz$ for some $|c| = 1$. Then $\varphi = c\psi$. Since $\varphi'(a) > 0, \psi'(a) > 0$, we must have $c = 1$. ☐

Definition. A Jordan curve is a continuous, one-to-one image of the set $\{z : |z| = 1\}$.

Theorem 13.0.14. Let G be a region bounded by a Jordan curve. Then G is simply connected, and any univalent function f that maps G onto D extends to a homeomorphism from G onto D.

Proof. See for instance Conway Volume 2, or books by Nehari, Nevanlinna-Paatera, and Burckel. ☐

Theorem 13.0.15 (Schwarz Reflection Principle)**.** Let f be analytic on a region G, and suppose that the boundary of G contains a free line segment L (where here L is free in the sense that for any point $z \in L$, there exists a disk centered at z such that half of the disk is contained in G). Then if f has a continuous extension to L, and if $f(L) \subset L'$ another line segment, then f has an analytic extension to $G \cup L \cup G'$, where G' is the reflection of G in L, such that if $z \in G$ and $z' \in G'$ are symmetric with respect to L, then their images are symmetric with respect to L'.

Proof. By applying appropriate linear fractional transformations we can reduce the theorem to the case when $L, L' \subset \mathbb{R}$. Apply Supplemental Problem #16 to solve this simpler case. ☐

Theorem 13.0.16. Let Π be a polygon with vertices $w_1, \ldots, w_n$ and interior angles $\pi\alpha_1$, $\ldots$, $\pi\alpha_n$. Then the univalent function $f : \mathbb{C}_+ \to \Pi$ mapping the upper half-plane onto the polygon Π, which also maps the points $a_1, \ldots, a_n \in \mathbb{R}$ to the vertices $w_1, \ldots, w_n$, respectively, is given by

$$f(z) = A\int_{z_0}^{z} \frac{d\zeta}{(\zeta - a_1)^{1-\alpha_1}\cdots(\zeta - a_n)^{1-\alpha_n}} + B$$

where $z_0 \in \mathbb{C}_+$ is any point in the upper half plane, A and B are constants that depend on the points $a_1, \ldots, a_n, w_1, \ldots, w_n$, and the integral is taken along any contour in $\mathbb{C}_+$ from z_0 to z.

Remark. By the Riemann Mapping Theorem, there exists a univalent function g mapping Π onto the unit disk D. We can choose a linear fractional transformation T such that T maps D onto $\mathbb{C}_+$, and T maps the points $g(w_1), \ldots, g(w_n)$ into the finite interval $(0, 2) \subset \mathbb{R}$. Then in the theorem above, $f = (T \circ g)^{-1}$.

Example (Instructive Preliminary Example)**.** Consider the function $g(z) = (z-a)^\alpha$, $a \in \mathbb{R}$, $0 < \alpha < 2\pi$ defined on $\mathbb{C}_+$. For $\alpha \neq 1$, g is defined by $g(z) = e^{\alpha \log(z-a)}$. Observe that g maps $\mathbb{C}_+$ onto a sector of the plane of angle $\pi\alpha$. Also, g can be extended as an analytic function across either $(-\infty, a)$ or across (a, ∞). Let g_1 and g_2 be the analytic extensions of g into $\mathbb{C}_-$ across these intervals. Then $g_1(z) = e^{\alpha \log(z-a)}$ with $\pi \leq \arg(z-a) < 2\pi$, and $g_2(z) = e^{\alpha \log(z-a)}$ with $-\pi < \arg(z-a) \leq 0$. Note that $\frac{g_1''}{g_1'} = \frac{g_2''}{g_2'} = \frac{\alpha-1}{z-a}$.

Proof. Let $f : \mathbb{C}_+ \to \Pi$ be the function in the theorem. The existence of f is guaranteed by the Riemann Mapping Theorem. By the Schwarz Reflection Principle, we can extend f across any of the subintervals $(a_i, a_{i+1}) \subset \mathbb{R}$, $1 \leq i \leq n-1$ (though there is no reason to expect that extensions across different intervals will be identical).

Consider the function $Q(z) = \frac{f''(z)}{f'(z)}$. Note that since f is univalent, f' is non-vanishing on $\mathbb{C}_+$, so $Q(z)$ is analytic on $\mathbb{C}_+$. We wish to show that Q extends to a function that is analytic on $\mathbb{C} \backslash \{a_1, \ldots, a_n\}$. Let f_1, f_2 be the extensions of f across two different intervals L_1, L_2 determined by the a_i. By the Schwarz Reflection Principle, the images $\Pi' = f_1(\mathbb{C}_-), \Pi'' = f_2(\mathbb{C}_-)$ are reflections of Π across edges of Π. Then Π'' is a rigid rotation/translation of Π', so for $z' \in \mathbb{C}_-$, we have $f_2(z') = Cf_1(z') + D$ for some constants $C, D \in \mathbb{C}$. Then $\frac{f_2''}{f_2'} = \frac{f_1''}{f_1'}$, and we conclude that we can extend Q uniquely to $\mathbb{C}_-$.

Now Q has isolated singularities at $a_1, \ldots, a_n$. Let $g(z) = c_j(z - w_j)^{1/\alpha_j}$, $|c_j| = 1$, and let $h(z) = g \circ f$. Then by the Schwarz Reflection Principle, h is analytic acorss an interval containing a_j and $h(a_j) = 0$. Then $h(z) = (z - a_j)h_1(z)$ for some analytic function $h_1(z)$ with $h_1(a_j) \neq 0$ (because $\operatorname{Im}(h(z)) > 0$ near $a_j \Rightarrow h_1(a_j) \neq 0$). Then $f(z) = f(a_j) + (z - a_j)^{\alpha_j} k(z)$ for some analytic function k with $k(a_j) \neq 0$. Now $\lim_{z \to a_j}(z - a_j)\frac{f''}{f'} = \alpha_j - 1$, so Q has a simple pole with residue $\alpha_j - 1$ at each a_j.

We have that $F(z) = Q(z) - \sum_{j=1}^n \frac{\alpha_j - 1}{z - a_j}$ is entire. The point ∞ is mapped by F to an edge of Π. By the Schwarz Reflection Principle, f is analytic at ∞. So $f(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$ for $|z| > r$. Then $\frac{f''}{f'} = O(z^{-1})$ as $z \to \infty$. Then by Liouville's Theorem, $F(z) \equiv 0$. Therefore $Q(z) = \frac{f''}{f'} = \sum_{j=1}^n \frac{\alpha_j - 1}{z - a_j}$. Now $\log f'(z) = \sum_{j=1}^n \log(z - a_j)^{\alpha_j - 1} + \text{constant}$.

Exponentiating, we have $f'(z) = A\frac{1}{(z-a_1)^{1-\alpha_1}\cdots(z-a_n)^{1-\alpha_n}}$ for some constant A, and integrating we have that

$$f(z) = A\int_{z_0}^{z} \frac{d\zeta}{(\zeta - a_1)^{1-\alpha_1}\cdots(\zeta - a_n)^{1-\alpha_n}} + B$$

for some constant B. □

Remark (Variants)**.**

1. If $a_n = \infty$, then that term does not appear in the formula for f.
2. If $w_n = \infty$, take $a_n = 0$.
3. We can take $z_0 \in \mathbb{R}$.

Example. Map $\mathbb{C}_+$ onto a triangle with interior angles $\pi\alpha, \pi\beta, \pi\gamma$. Map $0 = a_1 \to w_1$, $1 = a_2 \to w_2$, and $\infty = a_3 \to w_3$, where w_1, w_2, w_3 are the vertices with angles $\pi\alpha, \pi\beta, \pi\gamma$, respectively. And let a, b, c be the lengths of the sides opposite the angles $\pi\alpha, \pi\beta, \pi\gamma$, respectively. Then $f(z) = A\int_0^z \frac{d\zeta}{\zeta^{1-\alpha}(\zeta-1)^{1-\beta}} + B$. Take $A = 1, B = 0$. Then $c = \int_0^1 |f'(t)|\,dt = \int_0^1 \frac{dt}{t^{1-\alpha}(t-1)^{1-\beta}} = B(\alpha, \beta)$, the beta integral. Then $c = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$. Using the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we have that $c = \frac{1}{\pi}\sin(\pi z)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$. We can also use the Law of Sines $\frac{a}{\sin(\pi\alpha)} = \frac{b}{\sin(\pi\beta)} = \frac{c}{\sin(\pi\gamma)}$ to determine a and b.