

SINGULAR POLYNOMIALS FOR THE SYMMETRIC GROUPS

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ABSTRACT. For certain negative rational numbers κ_0 , called singular values, and associated with the symmetric group S_N on N objects, there exist homogeneous polynomials annihilated by each Dunkl operator when the parameter $\kappa = \kappa_0$. It was shown by the author, de Jeu and Opdam (Trans. Amer. Math. Soc. 346 (1994), 237-256) that the singular values are exactly the values $-\frac{m}{n}$ with $2 \leq n \leq N$, $m = 1, 2, \dots$ and $\frac{m}{n}$ is not an integer. This paper constructs for each pair (m, n) satisfying these conditions an irreducible S_N -module of singular polynomials for the singular value $-\frac{m}{n}$. The module is of isotype $(n-1, (n_1-1)^l, \rho)$ where $n_1 = n/\gcd(m, n)$, $\rho = N - (n-1) - l(n_1-1)$ and $1 \leq \rho \leq n_1-1$. The singular polynomials are special cases of nonsymmetric Jack polynomials. The paper presents some formulae for the action of Dunkl operators on these polynomials valid in general, and a method for showing the dependence of poles (in the parameter κ) on the number of variables. Murphy elements are used to analyze the representation of S_N on irreducible spaces of singular polynomials.

1. INTRODUCTION

We will construct polynomials on \mathbb{R}^N which are annihilated by each Dunkl operator associated with the symmetric group S_N , acting by permutation of coordinates, when the parameter takes on a singular value $-\frac{m}{n}$ with $2 \leq n \leq N$ and $-\frac{m}{n} \notin \mathbb{Z}$. The group S_N is considered as the finite reflection group of type A_{N-1} . Let \mathbb{N}_0 denote $\{0, 1, 2, 3, \dots\}$; for $\alpha \in \mathbb{N}_0^N$ (called a "composition") let $|\alpha| = \sum_{i=1}^N \alpha_i$ and define the monomial x^α to be $\prod_{i=1}^N x_i^{\alpha_i}$; its degree is $|\alpha|$. The length of a composition is $\ell(\alpha) = \max\{j : \alpha_j > 0\}$. For $1 \leq i \leq N$ let $\varepsilon(i) \in \mathbb{N}_0^N$ denote the standard basis element, that is, $\varepsilon(i)_j = \delta_{ij}$. Consider elements of S_N as functions on $\{1, 2, \dots, N\}$ then for $x \in \mathbb{R}^N$ and $w \in S_N$ let $(xw)_i = x_{w(i)}$ for $1 \leq i \leq N$; and extend this action to polynomials by $wf(x) = f(xw)$. This has the effect that monomials transform to monomials, $w(x^\alpha) = x^{w\alpha}$ where $(w\alpha)_i = \alpha_{w^{-1}(i)}$ for $\alpha \in \mathbb{N}_0^N$. (Consider x as a row vector, α as a column vector, and w as a permutation matrix, with 1's at the $(w(j), j)$ entries.) The reflections in S_N are the transpositions, denoted by (i, j) for $i \neq j$, interchanging x_i and x_j .

In [4] the author constructed for each finite reflection group a parametrized commutative algebra of differential-difference operators. Let κ be a formal parameter,

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that is, $\mathbb{Q}(\kappa)$ is a transcendental extension of \mathbb{Q} . For the symmetric group the operators are defined as follows:

Definition 1. For any polynomial f on \mathbb{R}^N and $1 \leq i \leq N$ let

$$\mathcal{D}_i f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j \neq i} \frac{f(x) - (ij) f(x)}{x_i - x_j}.$$

The polynomials under consideration are elements of $\text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^N\}$. It was shown in [4] that $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i$ for $1 \leq i, j \leq N$ and each \mathcal{D}_i maps homogeneous polynomials to homogeneous polynomials. A specific numerical parameter value κ_0 is said to be a *singular value* (associated with S_N) if there exists a nonzero polynomial p such that $\mathcal{D}_i p = 0$ for $1 \leq i \leq N$ when κ is specialized to κ_0 , and p is called a *singular polynomial*. It was shown in [7] that the singular values are the numbers $-\frac{j}{n}$ where $n = 2, \dots, N$, $j \in \mathbb{N}$ and $\frac{j}{n} \notin \mathbb{Z}$. The space of homogeneous polynomials of degree n , denoted by \mathcal{P}_n , is $\text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n\}$. The set of partitions of length $\leq N$ is denoted by $\mathbb{N}_0^{N,P}$ and consists of all $\lambda \in \mathbb{N}_0^N$ such that $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i \leq N-1$. When writing partitions it is customary to suppress trailing zeros and to use exponents to indicate multiplicity, for example $(5, 2^3)$ is the same as $(5, 2, 2, 2, 0) \in \mathbb{N}_0^{5,P}$. The irreducible representations of S_N are labeled by partitions of N (that is, $\tau \in \mathbb{N}_0^{N,P}$ and $|\tau| = N$) and we say a polynomial f is of *isotype* τ if f is an element of an irreducible S_N -submodule of \mathcal{P}_n on which the representation τ is realized. It was conjectured in [7] that the two-part representations $(\mu, N - \mu)$ (with $2\mu \geq N$) give rise to singular polynomials for the singular values $-\frac{m}{\mu+1}$ with $\gcd(m, \mu+1) < \frac{\mu+1}{N-\mu}$ (this was shown in [5]), and the representations $(s(\mu+1) - 1, \mu, \dots, \mu, \rho)$ for $s, \mu \in \mathbb{N}$ give rise to singular polynomials for the singular values $-\frac{m}{\mu+1}$ with $\gcd(m, \mu+1) = 1$. The latter is the main topic of this paper. For example, the singular values $-\frac{m}{6}$ for $N = 10$ are associated with the isotypes $(5, 5)$ for $m \equiv 1, 5 \pmod{6}$, $(5, 2, 2, 1)$ for $m \equiv 2, 4 \pmod{6}$, and $(5, 1^5)$ for $m \equiv 3 \pmod{6}$.

In the rest of this introduction we present definitions and key properties of nonsymmetric Jack polynomials, hook-length products for Ferrers diagrams, and the fundamental partial order on compositions. Section 2 contains detailed formulae for the action of $\{\mathcal{D}_i\}$ on the polynomials, with emphasis on the poles. The construction of singular polynomials is presented in Section 3, and there is a key result on the absence of certain poles when the number of variables (that is, N) is small enough. Murphy's construction [13] of the seminormal representations of S_N is used in Section 4 to analyze the irreducible S_N -modules generated by singular polynomials. The conclusion in Section 5 concisely displays the correspondence between pairs (m, n) , $2 \leq n \leq N$, $\frac{m}{n} \notin \mathbb{Z}$ and singular polynomials for $\kappa = -\frac{m}{n}$, and also considers modules of the specializations of the rational Cherednik algebra, defined in terms of singular polynomials.

Our construction will be in terms of nonsymmetric Jack polynomials. Since these have coefficients in $\mathbb{Q}(\kappa)$ with poles at negative rational values of κ it will be important to be precise about these poles. Any further reference to poles will be with respect to κ . The related commutative algebra of self-adjoint operators is

generated by

$$\mathcal{U}_i f(x) = \mathcal{D}_i x_i f(x) - \kappa \sum_{j=1}^{i-1} (j, i) f(x), 1 \leq i \leq N.$$

(this differs by an additive constant from the notation in [8, Ch.8]). The operators act in a triangular manner on monomials.

Definition 2. For $\alpha \in \mathbb{N}_0^N$ let α^+ denote the unique partition such that $\alpha^+ = w\alpha$ for some $w \in S_N$. For $\alpha, \beta \in \mathbb{N}_0^N$ the partial order $\alpha \succ \beta$ (α dominates β) means that $\alpha \neq \beta$ and $\sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i$ for $1 \leq j \leq N$; and $\alpha \triangleright \beta$ means that $|\alpha| = |\beta|$ and either $\alpha^+ \succ \beta^+$ or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$. The notations $\alpha \succeq \beta$ and $\alpha \supseteq \beta$ include the case that $\alpha = \beta$.

Acting on the monomial basis of \mathcal{P}_n the operators \mathcal{U}_i have on-diagonal coefficients involving the following ‘‘rank’’ function on \mathbb{N}_0^N . We denote the cardinality of a set E by $\#E$.

Definition 3. For $\alpha \in \mathbb{N}_0^N$ and $1 \leq i \leq N$ let

$$\begin{aligned} r(\alpha, i) &= \#\{j : \alpha_j > \alpha_i\} + \#\{j : 1 \leq j \leq i, \alpha_j = \alpha_i\}, \\ \xi_i(\alpha) &= (N - r(\alpha, i))\kappa + \alpha_i + 1. \end{aligned}$$

Clearly for a fixed $\alpha \in \mathbb{N}_0^N$ the values $\{r(\alpha, i) : 1 \leq i \leq N\}$ consist of all of $\{1, \dots, N\}$, are independent of trailing zeros (that is, if $\alpha' \in \mathbb{N}_0^M$, $\alpha'_i = \alpha_i$ for $1 \leq i \leq N$ and $\alpha'_i = 0$ for $N < i \leq M$ then $r(\alpha, i) = r(\alpha', i)$ for $1 \leq i \leq N$), and $\alpha \in \mathbb{N}_0^{N,P}$ if and only if $r(\alpha, i) = i$ for all i . Then (see [8, p.291]) $\mathcal{U}_i x^\alpha = \xi_i(\alpha) x^\alpha + q_{\alpha, i}(x)$ where $q_{\alpha, i}(x)$ is a sum of terms $\pm \kappa x^\beta$ with $\alpha \triangleright \beta$. The nonsymmetric Jack polynomials are the simultaneous eigenvectors of $\{\mathcal{U}_i : 1 \leq i \leq N\}$, well-defined for generic κ . Opdam [14, p.83] discovered and studied them in the wider framework of polynomials associated to crystallographic root systems. There are two useful normalizations of these polynomials, one is ‘‘monic in x ’’ the other is ‘‘monic in p ’’. The p -basis is defined by the generating function

$$\sum_{\alpha \in \mathbb{N}_0^N} p_\alpha(x) y^\alpha = \prod_{i=1}^N \left((1 - x_i y_i)^{-1} \prod_{j=1}^N (1 - x_i y_j)^{-\kappa} \right), \text{ for } \max_{i,j} |x_i| |y_j| < 1.$$

In contrast to the monomial basis $\mathcal{U}_i p_\alpha = \xi_i(\alpha) p_\alpha + q'_{\alpha, i}$ where $q'_{\alpha, i}$ is a sum of terms $\pm \kappa p_\beta$ with $\beta \triangleright \alpha$ (and $\ell(\beta) \leq \ell(\alpha)$), (see [8, Prop. 8.4.11]).

Definition 4. For $\alpha \in \mathbb{N}_0^N$ let $\zeta_\alpha, \zeta_\alpha^x$ denote the p -monic and x -monic, respectively, simultaneous eigenvectors, that is, $\mathcal{U}_i \zeta_\alpha = \xi_i(\alpha) \zeta_\alpha$, $\mathcal{U}_i \zeta_\alpha^x = \xi_i(\alpha) \zeta_\alpha^x$ for $1 \leq i \leq N$ and $\zeta_\alpha = p_\alpha + \sum_{\beta \triangleright \alpha} A_{\beta\alpha} p_\beta$, $\zeta_\alpha^x = x^\alpha + \sum_{\alpha \triangleright \beta} A_{\beta\alpha}^x x^\beta$, with coefficients $A_{\beta\alpha}, A_{\beta\alpha}^x \in \mathbb{Q}(\kappa)$.

Suppose that $\ell(\alpha) = m$ for some $m \geq 1$ then the coefficients $A_{\beta\alpha}$ do not depend on $N \geq m$ (and $A_{\beta\alpha} \neq 0$ implies $\ell(\beta) \leq m$); on the other hand, if $\beta \in \mathbb{N}_0^N$ and $\ell(\beta) \leq m$ then $A_{\beta\alpha}^x$ does not depend on $N \geq m$ (that is, if $N > M \geq m$ then the projection of \mathbb{R}^N onto \mathbb{R}^M setting $x_{M+1} = \dots = x_N = 0$ and annihilating the terms $A_{\beta\alpha}^x x^\beta$ with $\beta_i \neq 0$ for some $i > M$ ($\ell(\beta) > m$), produces the \mathbb{R}^M -polynomial. The relation between the two types involves hook-length products. Suppose $\lambda \in \mathbb{N}_0^{N,P}$ and $\ell(\lambda) = m$; the Ferrers diagram of λ is the set $\{(i, j) : 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$.

Each node (i, j) has the *arm* $\{(i, l) : j < l \leq \lambda_i\}$ and the *leg* $\{(l, j) : i < l, j \leq \lambda_l\}$. The node itself, the arm and the leg make up the *hook*. For $t \in \mathbb{Q}(\kappa)$ the *hook-length*, the hook-length product and generalized Pochhammer symbol for λ are given by

$$h(\lambda, t; i, j) = (\lambda_i - j + t + \kappa \# \{l : i < j, j \leq \lambda_l\})$$

$$h(\lambda, t) = \prod_{i=1}^m \prod_{j=1}^{\lambda_i} h(\lambda, t; i, j),$$

$$(t)_\lambda = \prod_{i=1}^m \prod_{j=1}^{\lambda_i} (t - (i-1)\kappa + j - 1).$$

The coordinate-wise notation for hook-lengths will appear in the context of specializations of κ to negative rational numbers. The compositions $\alpha \in \mathbb{N}_0^N$ are associated with the products

$$\mathcal{E}_\varepsilon(\alpha) = \prod \left\{ 1 + \frac{\varepsilon \kappa}{\kappa(r(\alpha, i) - r(\alpha, j)) + \alpha_j - \alpha_i} : i < j, \alpha_i < \alpha_j \right\}, \varepsilon = \pm.$$

Note that the denominator is identical to $\xi_j(\alpha) - \xi_i(\alpha)$ and $\mathcal{E}_\varepsilon(\lambda) = 1$ for $\lambda \in \mathbb{N}_0^{N,P}$. Then (see [8, p.323]) for each $\alpha \in \mathbb{N}_0^N$

$$\zeta_\alpha = \mathcal{E}_+(\alpha) \mathcal{E}_-(\alpha) \frac{h(\alpha^+, \kappa + 1)}{h(\alpha^+, 1)} \zeta_\alpha^x.$$

Also $\zeta_\alpha(1^N) = \mathcal{E}_-(\alpha) (N\kappa + 1)_{\alpha^+} / h(\alpha^+, 1)$. Knop and Sahi [11] by finding explicit combinatorial formulae in terms of tableaux established the key theorem that $h(\lambda, \kappa + 1) \zeta_\lambda^x$ is a polynomial with coefficients in $\mathbb{Z}[\kappa]$. However as N decreases the set of κ -poles of ζ_λ^x also decreases, and specific results will be established and used in the sequel (by [11, Cor. 4.7] the coefficient of $x_{m+1} \dots x_{m+n}$ in ζ_λ^x is $n! \kappa^n / h(\lambda, \kappa + 1)$ where $|\lambda| = n$ and $\ell(\lambda) = m$; thus for $m \leq N < m + n$ one expects some poles to be omitted). An obvious sufficient condition for the presence of a pole in ζ_λ^x for given N is its presence in $\zeta_\lambda^x(1^N) = (N\kappa + 1)_\lambda / h(\lambda, \kappa + 1)$.

We can now state our main results: for each isotype τ and singular value κ_0 the corresponding singular polynomials form the S_N -module generated by ζ_λ^x for a certain λ , that is, $\text{span}_{\mathbb{Q}}\{w\zeta_\lambda^x : w \in S_N\}$; (in fact a basis will be specified in terms of reverse lattice permutations of λ)

- for $\tau = (\mu, N - \mu)$, $\kappa_0 = -\frac{m}{\mu+1}$ with $\text{gcd}(m, \mu + 1) < \frac{\mu+1}{N-\mu}$, let $\lambda = (m^{N-\mu}, 0^\mu)$ (that is, m is repeated $N - \mu$ times followed by μ zeros)
- for $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$ where $l \geq 1, s \geq 0$, and $1 \leq \rho \leq \mu$ (so that $N = (s + l + 1)\mu + s + \rho$), $\kappa_0 = -\frac{m}{\mu+1}$ with $\text{gcd}(m, \mu + 1) = 1$, let $\lambda = ((m(s + l + 1))^\rho, (m(s + l))^\mu, \dots, (m(s + 1))^\mu, 0^{s(\mu+1)+\mu})$.

For example, let $N = 10, \tau = (5, 2, 2, 1), \kappa_0 = -\frac{m}{3}$ and $\text{gcd}(m, 3) = 1$, then $\lambda = (4m, 3m, 3m, 2m, 2m, 0^5)$. The singular polynomials for $N = 2k + 1, \tau = (2k - 1, 1, 1), \kappa_0 = -\frac{m}{2}, \lambda = (m(k + 1), mk)$ were found in [6] by a different method (not suitable for the general problem).

2. DIFFERENTIATION FORMULAE

This section contains expressions for $\mathcal{D}_i \zeta_\alpha$ in terms of $\{\zeta_\beta : |\beta| = |\alpha| - 1\}$, valid for generic κ . There is some material dealing with x -monic polynomials, however

the p -monic polynomials have somewhat simpler formulae. The basic step is the formula for $\mathcal{D}_m \zeta_\alpha$ where $\ell(\alpha) = m$, $\alpha \in \mathbb{N}_0^N$; further from properties of the p -basis it follows that $\mathcal{D}_i \zeta_\alpha = 0$ for $i > m$. The computation involves a cyclic shift. For $1 \leq i \leq j \leq N$ let $[i, j]$ denote the interval $\{k \in \mathbb{Z} : i \leq k \leq j\}$ and let $S[i, j]$ denote the subgroup of S_N generated by $\{(i, i+1), (i+1, i+2), \dots, (j-1, j)\}$ (isomorphic to S_{j+1-i}).

Definition 5. For $1 < m \leq N$ let $\theta_m = (1, 2)(2, 3) \dots (m-1, m) \in S_N$, and if $\alpha \in \mathbb{N}_0^N$ satisfies $\ell(\alpha) = m$ then $\tilde{\alpha} = \theta_m(\alpha - \varepsilon(m)) = (\alpha_m - 1, \alpha_1, \dots, \alpha_{m-1}, 0, \dots)$.

Lemma 1. Suppose $\alpha \in \mathbb{N}_0^N$ satisfies $\ell(\alpha) = m$ then (i) $\mathcal{U}_1 \theta_m \mathcal{D}_m \zeta_\alpha = (\xi_m(\alpha) - 1) \theta_m \mathcal{D}_m \zeta_\alpha$, (ii) $\mathcal{U}_i \theta_m \mathcal{D}_m \zeta_\alpha = \xi_{i-1}(\alpha) \theta_m \mathcal{D}_m \zeta_\alpha$ for $1 < i \leq m$, and (iii) $\mathcal{U}_i \theta_m \mathcal{D}_m \zeta_\alpha = ((N-i)\kappa + 1) \theta_m \mathcal{D}_m \zeta_\alpha$ for $i > m$.

Proof. The commutation $(x_m \mathcal{D}_m - \mathcal{D}_m x_m) f = -f - \kappa \sum_{j \neq m} (j, m) f$ ([8, p.290]) shows that

$$\begin{aligned} \mathcal{D}_m x_m \mathcal{D}_m \zeta_\alpha &= -\mathcal{D}_m \zeta_\alpha + \mathcal{D}_m \left(\mathcal{D}_m x_m - \kappa \sum_{j < m} (j, m) \right) \zeta_\alpha - \kappa \sum_{j > m} \mathcal{D}_m (j, m) \zeta_\alpha \\ &= \mathcal{D}_m (\xi_m(\alpha) - 1) \zeta_\alpha \end{aligned}$$

because $\mathcal{D}_m (j, m) \zeta_\alpha = (j, m) \mathcal{D}_j \zeta_\alpha = 0$ for $j > m$. Apply θ_m to the previous equation to prove part (i) (since $\theta_m \mathcal{D}_m x_m = \mathcal{D}_1 x_1 \theta_m$). Next suppose that $1 < i \leq m$ then $\theta_m^{-1} \mathcal{U}_i \theta_m = \mathcal{D}_{i-1} x_{i-1} - \kappa \sum_{j=1}^{i-2} (j, i-1) - \kappa(m, i-1)$. Apply this operator to $\mathcal{D}_m \zeta_\alpha$ to obtain

$$\begin{aligned} \theta_m^{-1} \mathcal{U}_i \theta_m \mathcal{D}_m \zeta_\alpha &= \mathcal{D}_m \left(\mathcal{D}_{i-1} x_{i-1} - \kappa \sum_{j=1}^{i-2} (j, i-1) \right) \zeta_\alpha \\ &\quad + \kappa (\mathcal{D}_{i-1} (i-1, m) - (m, i-1) \mathcal{D}_m) \\ &= \mathcal{D}_m \xi_{i-1}(\alpha) \zeta_\alpha. \end{aligned}$$

The computation uses the commutativity of \mathcal{D}_m and \mathcal{D}_{i-1} and the commutation $(x_j \mathcal{D}_m - \mathcal{D}_m x_j) f = \kappa(j, m) f$ ([8, p.290]) for $j \neq m$. This shows part (ii). Similarly for $i > m$ we have that $\theta_m^{-1} \mathcal{U}_i \theta_m = \mathcal{U}_i$ and $\mathcal{U}_i \mathcal{D}_m - \mathcal{D}_m \mathcal{U}_i = \kappa(\mathcal{D}_i(i, m) - (i, m) \mathcal{D}_m + \mathcal{D}_m(i, m)) = \kappa(i, m) \mathcal{D}_i$. But $\mathcal{D}_i \zeta_\alpha = 0$ for $i > m$ and so $\mathcal{U}_i \mathcal{D}_m \zeta_\alpha = \xi_i(\alpha) \mathcal{D}_m \zeta_\alpha$ and $\xi_i(\alpha) = (N-i)\kappa + 1$; proving part (iii). \square

The following is used to pick out a coefficient in $\mathcal{D}_m \zeta_\alpha$.

Lemma 2. Suppose $\alpha, \beta \in \mathbb{N}_0^N$, $|\alpha| = |\beta|$ and $\ell(\alpha) = \ell(\beta) = m$, if $p_{\alpha - \varepsilon(m)}$ appears with a nonzero coefficient in the expansion of $\mathcal{D}_m p_\beta$ then either $\alpha = \beta$ or $\beta \triangleleft \alpha$. If $\alpha = \beta$ then the coefficient is $(N+1 - r(\alpha, m))\kappa + \alpha_m$.

Proof. By Prop.8.4.3 [8, p.294]

$$\begin{aligned} \mathcal{D}_m p_\beta &= ((N - \#\{j : \beta_j \geq \beta_m\} + 1)\kappa + \beta_m) p_{\beta - \varepsilon(m)} \\ &\quad + \kappa \sum \{p_\gamma : \gamma = \beta + n\varepsilon(m) - (n+1)\varepsilon(j), \max(0, \beta_j - \beta_m) \leq n \leq \beta_j - 1, j \neq m\} \\ &\quad - \kappa \sum \{p_\gamma : \gamma = \beta - (n+1)\varepsilon(m) + n\varepsilon(j), \max(1, \beta_m - \beta_j) \leq n \leq \beta_m - 1, j \neq m\}. \end{aligned}$$

If $\beta = \alpha$ then the coefficient of $p_{\alpha - \varepsilon(m)}$ is $(N - r(\alpha, m) + 1)\kappa + \alpha_m$; note that $j > m$ implies $\alpha_j = 0 < \alpha_m$. If $p_{\alpha - \varepsilon(m)}$ has the coefficient κ then $\beta = \alpha -$

$(n+1)(\varepsilon(m) - \varepsilon(j))$ and the restrictions are equivalent to $0 \leq n \leq \alpha_m - \alpha_j - 2$ (for some $j \neq m$); thus $n+1 < \alpha_m - \alpha_j$ and $\beta^+ \prec \alpha^+$ by Lemma 8.2.3(iv) [8, p.289]. If $p_{\alpha-\varepsilon(m)}$ has the coefficient $-\kappa$ then $\beta = \alpha - n(\varepsilon(j) - \varepsilon(m))$ and the restrictions are equivalent to $1 \leq n \leq \alpha_j - \alpha_m$ (for some $j \neq m$). If $n < \alpha_j - \alpha_m$ then by the same lemma $\beta^+ \prec \alpha^+$. If $n = \alpha_j - \alpha_m$ then $\beta = (j, m)\alpha$ and $\beta \prec \alpha$ because $\alpha_j > \alpha_m$ and $j < m$ (using the hypothesis $\ell(\alpha) = m$). Thus, if $\alpha \neq \beta$ then $\beta \triangleleft \alpha$. \square

Theorem 1. *Suppose $\alpha \in \mathbb{N}_0^N$ and $\ell(\alpha) = m$, then*

$$\mathcal{D}_m \zeta_\alpha = ((N+1 - r(\alpha, m))\kappa + \alpha_m)\theta_m^{-1}\zeta_{\tilde{\alpha}}.$$

Proof. By Lemma 1, $\theta_m \mathcal{D}_m \zeta_\alpha$ is a simultaneous eigenvector of $\{\mathcal{U}_i : 1 \leq i \leq N\}$ with eigenvalues $(\xi_m(\alpha) - 1, \xi_1(\alpha), \dots, \xi_{m-1}(\alpha), \xi_{m+1}(\alpha), \dots)$. We claim these are the eigenvalues for $\tilde{\alpha}$. Indeed $r(\tilde{\alpha}, 1) = \#\{j : \alpha_j > \alpha_m - 1, j < m\} + 1 = \#\{j : \alpha_j \geq \alpha_m\} = r(\alpha, m)$. For any values α_i different from $\alpha_m - 1$ and 0 it is obvious that $r(\alpha, i) = r(\tilde{\alpha}, i+1)$. Suppose for some $i < m$ that $\alpha_i = \alpha_m - 1$ then

$$\begin{aligned} r(\tilde{\alpha}, i+1) &= \#\{j : \alpha_j > \alpha_m - 1, j < m\} + \#\{j : j \leq i, \alpha_j = \alpha_m - 1\} + 1 \\ &= \#\{j : \alpha_j > \alpha_m - 1, j \leq m\} + \#\{j : j \leq i, \alpha_j = \alpha_m - 1\} \\ &= r(\alpha, i). \end{aligned}$$

For $i > m$, obviously $\xi_i(\alpha) = \xi_i(\tilde{\alpha}) = (N-i)\kappa + 1$. Thus $\theta_m \mathcal{D}_m \zeta_\alpha = c\zeta_{\tilde{\alpha}}$ for some constant c , which will be determined by finding the coefficient of $\theta_m^{-1}p_{\tilde{\alpha}} = p_{\alpha-\varepsilon(m)}$ in $\mathcal{D}_m \zeta_\alpha$. Since $\zeta_\alpha = p_\alpha + \sum_{\beta \triangleright \alpha} A_{\beta\alpha} p_\beta$ (and $\ell(\beta) \leq m$) we obtain $\mathcal{D}_m \zeta_\alpha = \mathcal{D}_m p_\alpha + \sum_{\beta \triangleright \alpha} A_{\beta\alpha} \mathcal{D}_m p_\beta$. If $p_{\alpha-\varepsilon(m)}$ has a nonzero coefficient in $\mathcal{D}_m p_\beta$ then $\ell(\beta) = m$ (else $\beta_m = 0$ and $\mathcal{D}_m p_\beta = 0$) and by Lemma 2 $\beta = \alpha$ or $\beta \triangleright \alpha$. Only the case $\beta = \alpha$ can occur and thus $c = (N+1 - r(\alpha, m))\kappa + \alpha_m$. \square

With the intention of using the theorem to compute arbitrary $\mathcal{D}_i \zeta_\lambda$ with $\lambda \in \mathbb{N}_0^{N,P}$ we observe that it suffices to consider the points of decrease, that is, $\lambda_i > \lambda_{i+1}$, (the values of i for which $\lambda - \varepsilon(i)$ is a partition) then apply the transpositions $(j, j+1)$ successively for $j = i, i+1, \dots, \ell(\lambda) - 1$, apply \mathcal{D}_m , with a result involving ζ_α where $\alpha = (\lambda_i - 1, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots)$ (this is an over-simplification; actually all the points of decrease between i and $\ell(\lambda)$ must be considered). Finally transform $\zeta_{\lambda-\varepsilon(i)}$ to ζ_α with another sequence of transpositions. As mentioned before it is necessary to keep track of the κ -poles occurring in these operations. The basic step is the action of an adjacent transposition on ζ_α .

Proposition 1. *Suppose $\alpha \in \mathbb{N}_0^N$, and $\alpha_i > \alpha_{i+1}$ for some i , then let $\sigma = (i, i+1)$ and $a = \kappa((r(\alpha, i+1) - r(\alpha, i))\kappa + \alpha_i - \alpha_{i+1})^{-1}$ then $\zeta_{\sigma\alpha} = \sigma\zeta_\alpha - a\zeta_\alpha$ and $\zeta_{\sigma\alpha}^x = \frac{1}{1-a^2}(\sigma\zeta_\alpha^x - a\zeta_\alpha^x)$.*

The proof for the p -monic case is in Prop. 8.5.5 [8, p.301]; the proof for the x -monic case can be deduced from the inverse of the p -monic formula and the equation $\zeta_\alpha^x = \sigma\zeta_{\sigma\alpha}^x + a\zeta_\alpha^x$ (arguing that x^α does not appear in $\zeta_{\sigma\alpha}^x$ since $\alpha \triangleright \sigma\alpha$). Note that the denominator $(r(\alpha, i+1) - r(\alpha, i))\kappa + \alpha_i - \alpha_{i+1} = \xi_i(\alpha) - \xi_{i+1}(\alpha)$. For singular values of κ it can happen that $a = -1$ implying that $\sigma\zeta_\alpha^x = -\zeta_\alpha^x$. We need an extension of the proposition applying to the situation of several adjacent entries of α being equal.

Proposition 2. Suppose $\alpha \in \mathbb{N}_0^N$ with $\alpha_i = a > b = \alpha_{i+j}$ for $1 \leq j \leq s$, where $1 \leq i < i + s \leq N$, then

$$\zeta_{(i,i+s)\alpha} = (i, i + s) \zeta_\alpha - \frac{\kappa}{(r(\alpha, i + s) - r(\alpha, i)) \kappa + a - b} \left(1 + \sum_{j=1}^{s-1} (i, i + j) \right) \zeta_\alpha.$$

Proof. Observe that $(i, i + s)\alpha = (\dots, b, \dots, b, a, \dots)$. The proof is by induction on s and depends on the invariance of ζ_α under the subgroup $S[i + 1, i + s]$. Since $r(\alpha, i + j) = r(\alpha, i + 1) + j - 1$ for $1 \leq j \leq s$ let $C = (r(\alpha, i + 1) - r(\alpha, i) - 1) \kappa + a - b$ so that $(r(\alpha, i + j) - r(\alpha, i)) \kappa + a - b = j\kappa + C$, also let $f_j = \zeta_{(i,i+j)\alpha}$ and $c_j = -\frac{\kappa}{j\kappa + C}$. By Proposition 1 $f_{j+1} = (i + j, i + j + 1) f_j + c_{j+1} f_j$. By the inductive hypothesis $f_j = (i, i + j) \zeta_\alpha + c_j \left(1 + \sum_{k=1}^{j-1} (i, i + k) \right) \zeta_\alpha$. Thus

$$\begin{aligned} f_{j+1} &= (i + j, i + j + 1) (i, i + j) \zeta_\alpha + c_{j+1} (i, i + j) \zeta_\alpha \\ &\quad + c_j ((i + j, i + j + 1) + c_{j+1}) \left(1 + \sum_{k=1}^{j-1} (i, i + k) \right) \zeta_\alpha \\ &= (i, i + j) \zeta_\alpha + c_{j+1} (i, i + j) \zeta_\alpha \\ &\quad + c_j \left(1 + \sum_{k=1}^{j-1} (i, i + k) \right) ((i + j, i + j + 1) + c_{j+1}) \zeta_\alpha \\ &= (i, i + j) \zeta_\alpha + c_{j+1} (i, i + j) \zeta_\alpha + c_j (1 + c_{j+1}) \left(1 + \sum_{k=1}^{j-1} (i, i + k) \right) \zeta_\alpha. \end{aligned}$$

By the invariance property of ζ_α we have $(i + j, i + j + 1) (i, i + j) \zeta_\alpha = (i + j, i + j + 1) (i, i + j) (i + j, i + j + 1) \zeta_\alpha = (i, i + j) \zeta_\alpha$. Furthermore $c_j (1 + c_{j+1}) = -\frac{\kappa}{j\kappa + C} \left(1 - \frac{\kappa}{j\kappa + \kappa + C} \right) = c_{j+1}$ and this completes the induction. \square

There is a similar result for the opposite direction.

Proposition 3. Suppose $\alpha \in \mathbb{N}_0^N$ with $\alpha_{i+j} = b > a = \alpha_{i+s}$ for $0 \leq j \leq s - 1$, where $1 \leq i < i + s \leq N$, then

$$\zeta_{(i,i+s)\alpha} = (i, i + s) \zeta_\alpha - \frac{\kappa}{(r(\alpha, i + s) - r(\alpha, i)) \kappa + b - a} \left(1 + \sum_{j=1}^{s-1} (i + j, i + s) \right) \zeta_\alpha.$$

Proof. Proceeding similarly to the previous case, for $1 \leq j \leq s$, $r(\alpha, i + s - j) = r(\alpha, i + s - 1) + 1 - j$ and let $f_j = \zeta_{(i+i+s-j,i+s)\alpha}$ and $c_j = -\kappa ((r(\alpha, i + s) - r(\alpha, i + s - 1) - 1 + j) \kappa + b - a)^{-1}$. Also ζ_α is invariant under $S[i, i + s - 1]$. The inductive step is based on $f_{j+1} = (i + s - j - 1, i + s - j) f_j + c_{j+1} f_j$ (and $f_0 = \zeta_\alpha$). The rest of the argument is similar to the previous one and is omitted. \square

To illustrate the basic step, apply \mathcal{D}_{i+s} to both sides of the formula in Proposition 2 and obtain

$$\begin{aligned} \mathcal{D}_{i+s}\zeta_{(i,i+s)\alpha} &= (i, i+s) \mathcal{D}_i\zeta_\alpha \\ &- \frac{\kappa}{(r(\alpha, i+s) - r(\alpha, i))\kappa + a - b} \left(1 + \sum_{j=1}^{s-1} (i, i+j) \right) \mathcal{D}_{i+s}\zeta_\alpha, \\ \mathcal{D}_i\zeta_\alpha &= (i, i+s) \mathcal{D}_{i+s}\zeta_{(i,i+s)\alpha} \\ &+ \frac{\kappa}{(r(\alpha, i+s) - r(\alpha, i))\kappa + a - b} (i, i+s) \left(1 + \sum_{j=1}^{s-1} (i, i+j) \right) \mathcal{D}_{i+s}\zeta_\alpha; \end{aligned}$$

so that the index for \mathcal{D} is increased (eventually to m).

Fix a partition $\lambda \in \mathbb{N}_0^{N,P}$, suppose that the parts of λ have M distinct nonzero values, the points of decrease are $i_1 < i_2 < \dots < i_M$, so that λ_i is constant on each interval $i_{j-1} < i \leq i_j$ (interpret $i_0 = 0$, also let $\ell(\lambda) = m = i_M$). For $1 \leq j < k \leq M$ let

$$\begin{aligned} C_{jk} &= \frac{\kappa}{(i_k - i_j)\kappa + \lambda_{i_j} - \lambda_{i_k}}, \\ w_j &= 1 + \sum_{r=i_j+1}^{i_{j+1}-1} (i_j, r) \in \mathbb{ZS}[i_j, i_{j+1} - 1], \\ z_{jk} &= (i_{k-1}, i_k) - C_{jk}w_{k-1}, \end{aligned}$$

further let $\mu(j, k) \in \mathbb{N}_0^N$ be the action on λ by the cyclic shift on the interval $\{i_j, \dots, i_k\}$, that is $\mu(j, k)_{i_k} = \lambda_{i_j}$, $\mu(j, k)_i = \lambda_{i+1}$ for $i_j \leq i < i_k$ and $\mu(j, k)_i = \lambda_i$ for $i < i_j$ or $i > i_k$. Proposition 2 applies to the transformation of $\zeta_{\mu(j,k)}$ to $\zeta_{\mu(j,k+1)}$; note that $r(\mu(j, k), i_k) = i_j$ and $r(\mu(j, k), i_{k+1}) = i_{k+1}$ thus $\zeta_{\mu(j,k+1)} = z_{j,k+1} \zeta_{\mu(j,k)}$. The start of this recurrence is $\zeta_{\mu(j,j)} = \zeta_\lambda$. The object is to express any $\mathcal{D}_i\zeta_\lambda$ in terms of $\mathcal{D}_m\zeta_{\mu(j,M)}$, $j = 1, \dots, M$. It suffices to consider $\{\mathcal{D}_{i_j}\zeta_\lambda\}$ since $\mathcal{D}_i\zeta_\lambda = (i, i_j) \mathcal{D}_{i_j}\zeta_\lambda$ for $i_{j-1} < i < i_j$.

Lemma 3. For $k = 1, \dots, M - j$

$$\begin{aligned} \mathcal{D}_{i_M} z_{j,M} \dots z_{j,M-k+1} &= (i_{M-1}, i_M) (i_{M-2}, i_{M-1}) \dots (i_{M-k}, i_{M-k+1}) \mathcal{D}_{i_{M-k}} \\ &- \sum_{s=0}^{k-1} C_{j,M-s} (i_{M-1}, i_M) \dots (i_{M-s}, i_{M-s+1}) w_{M-s-1} z_{j,M-s-1} \dots z_{j,M-k+1} \mathcal{D}_{i_{M-s}}. \end{aligned}$$

Proof. We proceed by induction. The formula is tautological for $k = 0$. Also the term in the sum with $s = k - 1$ has no $z_{j,n}$ factors. Multiply the right hand side by $z_{j,M-k}$ on the right. For the first part, $\mathcal{D}_{i_{M-k}} ((i_{M-k-1}, i_{M-k}) - C_{j,M-k} w_{M-k-1}) = (i_{M-k-1}, i_{M-k}) \mathcal{D}_{i_{M-k-1}} - C_{j,M-k} w_{M-k-1} \mathcal{D}_{i_{M-k}}$ (since \mathcal{D}_{i_n} commutes with w_j for $n \neq j$). For the second part, $\mathcal{D}_{i_{M-s}}$ commutes with $z_{j,M-k}$. This completes the induction. \square

Set $k = M - j$ in the lemma, apply the operator to ζ_λ and multiply both sides of the identity by $(i_j, i_{j+1}) \dots (i_{M-1}, i_M)$, yielding (replace s by $M - s$)

$$\begin{aligned} \mathcal{D}_{i_j} \zeta_\lambda &= (i_j, i_{j+1}) \dots (i_{M-1}, i_M) \mathcal{D}_m \zeta_{\mu(j, M)} \\ &+ \sum_{s=j+1}^M C_{j, s} (i_j, i_{j+1}) \dots (i_{s-1}, i_s) w_{s-1} z_{j, s-1} \dots z_{j, j+1} \mathcal{D}_{i_s} \zeta_\lambda. \end{aligned}$$

This identity is used starting with $j = M - 1$ and then decrementing j by 1 with the result:

$$\mathcal{D}_{i_j} \zeta_\lambda = (i_j, i_{j+1}) \dots (i_{M-1}, i_M) \mathcal{D}_m \zeta_{\mu(j, M)} + \sum_{s=j+1}^M u_{j, s} \mathcal{D}_m \zeta_{\mu(s, M)},$$

where each $u_{j, s} \in RS[i_1, m]$ and R is the \mathbb{Z} -ring generated by $\{C_{j, k} : 1 \leq j < k \leq M\}$. To complete the analysis of $\mathcal{D}_m \zeta_{\mu(j, M)}$, for $0 \leq k < j$ let $\nu(k, j) \in \mathbb{N}_0^N$ be the action on $\lambda - \varepsilon(i_j)$ by the (reverse) cyclic shift on the interval $\{i_k + 1, \dots, i_j\}$, that is $\nu(k, j)_{i_{k+1}} = \lambda_{i_j} - 1$, $\nu(k, j)_i = \lambda_{i-1}$ for $i_k + 1 < i \leq i_j$ and $\nu(k, j)_i = \lambda_i$ for $i \leq i_k$ or $i > i_j$. Also let $\nu(j, j) = \lambda - \varepsilon(i_j)$; if $i_j = i_{j-1} + 1$ then $\nu(j-1, j) = \nu(j, j)$. For $0 \leq k < j-1 < M$ let

$$\begin{aligned} C'_{kj} &= \frac{\kappa}{(i_j - i_k - 1) \kappa + \lambda_{i_{k+1}} - \lambda_{i_j} + 1}, \\ w'_k &= 1 + \sum_{r=i_k+2}^{i_{k+1}} (r, i_{k+1} + 1) \in \mathbb{Z}S[i_k + 2, i_{k+1} + 1], \end{aligned}$$

and if $i_{j-1} < i_j - 1$ let

$$\begin{aligned} C'_{j-1, j} &= \frac{\kappa}{(i_j - i_{j-1} - 1) \kappa + 1}, \\ w'_{j-1} &= 1 + \sum_{r=i_{j-1}+2}^{i_j-1} (r, i_j) \in \mathbb{Z}S[i_{j-1} + 2, i_j]. \end{aligned}$$

Proposition 3 applies to the transformation of $\zeta_{\nu(k, j)}$ to $\zeta_{\nu(k-1, j)}$; note that $r(\nu(k, j), i_k + 1) = i_k + 1$ and $r(\nu(k, j), i_{k+1}) = i_j$. Thus

$$\zeta_{\nu(j-1, j)} = ((i_{j-1} + 1, i_j) - C'_{j-1, j} w'_{j-1}) \zeta_{\nu(j, j)}$$

(unless $i_j = i_{j-1} + 1$ when $\zeta_{\nu(j-1, j)} = \zeta_{\nu(j, j)}$) and

$$\zeta_{\nu(k, j)} = ((i_k + 1, i_{k+1} + 1) - C'_{kj} w'_k) \zeta_{\nu(k+1, j)}$$

for $0 \leq k \leq j-2$. By Theorem 2 $\mathcal{D}_m \zeta_{\mu(j, M)} = \theta_m^{-1} \zeta_{\tilde{\mu}(j, M)}$ where $\tilde{\mu}(j, M) = \nu(0, j)$ (that is, first the i_j -entry of λ is moved to the m -entry at the end, the action of \mathcal{D}_m decrements λ_{i_j} by 1 and moves it to the front, loosely speaking). In turn $\zeta_{\tilde{\mu}(j, M)}$ can be expressed in terms of $\zeta_{\nu(j, j)}$. The following is now established.

Theorem 2. Suppose $\lambda \in \mathbb{N}_0^{N, P}$ with points of increase $i_1 < i_2 < \dots < i_M = \ell(\lambda)$, let R be the \mathbb{Z} -ring generated by $\{C_{jk} : 1 \leq j < k \leq M\} \cup \{C'_{kj} : 0 \leq k < j \leq M\} \cup \mathbb{Z}$ and let $\lambda^{(j)} = \lambda - \varepsilon(i_j) \in \mathbb{N}_0^{N, P}$, then for $i_{j-1} < i \leq i_j$ with $0 \leq j \leq M$,

$$\mathcal{D}_i \zeta_\lambda = \sum_{s=j}^M ((N + 1 - i_s) \kappa + \lambda_{i_s}) u_{is} \zeta_{\lambda^{(s)}},$$

where each $u_{is} \in RS[1, m]$.

The Theorem exhibits the poles in the differentiation formula for the p -monic polynomials. To convert this for use with x -monic polynomials multiply $\zeta_{\lambda^{(s)}}$ by $(h(\lambda, 1)h(\lambda^{(s)}, \kappa + 1)) / (h(\lambda^{(s)}, 1)h(\lambda, \kappa + 1))$, then the identity holds for ζ replaced by ζ^x . The details are not worked out since in general there is no significant simplification. In the next section this calculation will be carried out for the singular polynomials.

3. EXISTENCE OF SINGULAR POLYNOMIALS

In this section we will show for certain $\lambda \in \mathbb{N}_0^{N,P}$ and singular values κ_0 that ζ_λ^x has no poles at $\kappa = \kappa_0$ and that $\mathcal{D}_i \zeta_\lambda^x = 0$ for $1 \leq i \leq N$. It turns out that for $m = \ell(\lambda)$ the last coefficient in the formula of Theorem 2 satisfies $(N + 1 - m)\kappa_0 + \lambda_m = 0$ and $\zeta_{\lambda - \varepsilon(m)}$ has no poles at κ_0 in general. For the terms of type $\zeta_{\lambda^{(s)}}^x$ the denominator expression $h(\lambda^{(s)}, \kappa + 1)$ has a zero at $\kappa = \kappa_0$ but the pole $(\kappa - \kappa_0)$ does not appear for the restriction to \mathbb{R}^N , and this is the key fact. We start with the isotypes of two-part partitions ($\tau = (\mu, N - \mu)$).

Proposition 4. *Let $\frac{N}{2} \leq \mu < N$, $\gcd(m, \mu + 1) < \frac{\mu + 1}{N - \mu}$, $\lambda = (m^{N - \mu})$ then $h(\lambda, 1)$, $h(\lambda, \kappa + 1)$, $h(\lambda - \varepsilon(N - \mu), 1)$ and $h(\lambda - \varepsilon(N - \mu), \kappa + 1)$ are nonzero when evaluated at $\kappa = -\frac{m}{\mu + 1}$.*

Proof. For $1 \leq i \leq N - \mu$, $1 \leq j \leq m$ we have $h(\lambda, t; i, j) = m - j + t + (N - \mu - i)\kappa$. For $t = 1, \kappa + 1$ the sets of values are $\{i\kappa + j : 0 \leq i \leq N - \mu - 1, 1 \leq j \leq m\}$, $\{i\kappa + j : 1 \leq i \leq N - \mu, 1 \leq j \leq m\}$ respectively. It suffices to show that the second set does not contain 0 for $\kappa = -\frac{m}{\mu + 1}$. Suppose $-im + j(\mu + 1) = 0$ for some (nonzero) i, j and let $d = \gcd(m, \mu + 1)$, then $\frac{\mu + 1}{d} | i \leq N - \mu$ which implies $\frac{\mu + 1}{N - \mu} \leq d$, contrary to the hypothesis. For $\lambda - \varepsilon(N - \mu)$ only the hook-lengths in the last row and column change; $h(\lambda - \varepsilon(N - \mu), t; i, m) = t + (N - \mu - i)\kappa$ and $h(\lambda - \varepsilon(N - \mu), t; N - \mu, j) = m - j + t$ for $1 \leq i < N - \mu$ and $1 \leq j < m$. These values have already been shown to be nonzero for $\kappa = -\frac{m}{\mu + 1}$. \square

Next we handle the case of three or more parts, for the isotype $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$. The following is the central hypothesis for this section.

Definition 6. *For $\mu, l \geq 1, s \geq 0, 1 \leq \rho \leq \mu$ and $\gcd(m, \mu + 1) = 1$ let*

$$\Lambda(\mu, s, l, \rho, m) = ((m(s + l + 1))^\rho, (m(s + l))^\mu, \dots, (m(s + 1))^\mu),$$

a partition of length $l\mu + \rho$ which is associated to the singular value $\kappa_0 = -\frac{m}{\mu + 1}$ and the S_N -representation of isotype $(s(\mu + 1) + \mu, \mu^l, \rho)$, where $N = (s + l + 1)\mu + s + \rho$.

Lemma 4. *Suppose $a, b, c \in \mathbb{N}_0$ and $c \geq 1, b \leq \mu$ then $a(\mu\kappa + m) + b\kappa + c \neq 0$ for $\kappa = -\frac{m}{\mu + 1}$ (where $\gcd(m, \mu + 1) = 1$).*

Proof. Denote the value of the expression at $\kappa = -\frac{m}{\mu + 1}$ by v , then $v = (a - b)\frac{m}{\mu + 1} + c$. If $a \geq b$ then $v \geq c \geq 1$; otherwise $0 > a - b \geq -\mu$ and $\mu + 1$ does not divide $(a - b)m$ thus $v \notin \mathbb{Z}$ and $v \neq 0$. \square

Proposition 5. *Let $\lambda = \Lambda(\mu, s, l, \rho, m)$, then $h(\lambda, 1)$ and $h(\lambda, \kappa + 1)$ are nonzero when $\kappa = -\frac{m}{\mu + 1}$.*

Proof. Since hook-lengths in a given row depend only on it and the rows of higher index we may assume that $\rho = \mu$. We index the rows of λ by $\mu k + i$ with $0 \leq k \leq l$ and $1 \leq i \leq \mu$, and the columns by $m(s+n) - j$ where $1 \leq n \leq l+1-k$ and $0 \leq j \leq m-1$ if $n > 1$, or $0 \leq j < m(s+1)$ if $n = 1$. Then

$$\begin{aligned} h(\lambda, t; \mu k + i, m(s+n) - j) &= \kappa((l+2-k-n)\mu - i) + m(l+1-k-n) + j + t \\ &= (l+1-k-n)(\kappa\mu + m) + \kappa(\mu - i) + j + t. \end{aligned}$$

Set $i' = \mu - i + 1$ if $t = \kappa + 1$ or $i' = \mu - i$ if $t = 1$; then the above expression equals $(l+1-k-n)(\kappa\mu + m) + \kappa i' + j + 1$ which is nonzero at $\kappa = \kappa_0$ by Lemma 4 (since $i' \leq \mu$). \square

Next we consider the hook-lengths for $\Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k\mu)$ with $0 \leq k \leq l$.

Proposition 6. *For $0 \leq k_0 \leq l$ let $\nu = \Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k_0\mu)$, then for $\kappa = -\frac{m}{\mu+1}$ $h(\nu, 1)$ is nonzero and $h(\nu, \kappa + 1)$ is nonzero for $k_0 = l$ and has a zero of multiplicity one for $0 \leq k_0 < l$, in the hook-length $h(\nu, \kappa + 1; \rho + k_0\mu, m(s+l+1-k_0))$.*

Proof. As in the previous proof, assume $\rho = \mu$. The column above the node deleted from λ (namely, $m(s+l+1-k_0)$) meets the rows labeled by $\mu k + i$ with $0 \leq k \leq k_0$ and $1 \leq i \leq \mu$, except $1 \leq i < \mu$ when $k = k_0$. Then $h(\nu, t; \mu k + i, m(s+l+1-k_0)) = (k_0 - k)(\kappa\mu + m) + \kappa i' + 1$ where $i' = \mu - 1 - i$ for $t = 1$ and $i' = \mu - i$ for $t = \kappa + 1$. By Lemma 4 the value is nonzero for $\kappa = \kappa_0$. The row of the deleted node meets the columns labeled $m(s+n) - j$ with $1 \leq n \leq l+1-k_0$. Then $h(\nu, t; m(s+n) - j, (k_0 + 1)\mu) = (l+1-k_0-n)(\kappa\mu + m) + b\kappa + j$, where $b = 0$ for $t = 1$ and $b = 1$ for $t = \kappa + 1$. The Lemma applies unless $j = 0$. Suppose $j = 0$ then $1 \leq n \leq l-k_0$ (the value $j = 0$ does not occur for $n = l+1-k_0$ since the corresponding node was deleted); at $\kappa = \kappa_0$ the value of the hook-length is $(l+1-k_0-n-b)\frac{m}{\mu+1}$ which is zero exactly when $n = l-k_0$ and $b = 1$ (that is, $t = \kappa + 1$). Thus the hook-length $h(\nu, \kappa + 1; m(s+l-k_0), (k_0 + 1)\mu) = \kappa(\mu + 1) + m$ is the only zero in $h(\nu, \kappa + 1)$ at $\kappa = \kappa_0$. \square

Next we show that the coefficients C_{jk} and C'_{kj} appearing in Theorem 2 have no poles at $\kappa = \kappa_0$. The points of decrease of $\Lambda(\mu, s, l, \rho, m)$ are $i_j = \rho + (j-1)\mu$, $\lambda_{i_j} = m(s+2+l-j)$ for $1 \leq j \leq l+1$. For $1 \leq j < k \leq l+1$ the coefficient $C_{jk} = \frac{\kappa}{(i_k - i_j)\kappa + \lambda_{i_j} - \lambda_{i_k}} = \frac{\kappa}{(k-j)(\kappa\mu + m)}$, which has value $-\frac{1}{k-j}$ at $\kappa = \kappa_0$.

Proposition 7. *For $\lambda = \Lambda(\mu, s, l, \rho, m)$ the coefficients C'_{kj} have no poles at $\kappa = -\frac{m}{\mu+1}$ for $0 \leq k < j \leq l+1$.*

Proof. First the special cases $C'_{j-1,j} = \frac{\kappa}{(\mu-1)\kappa+1}$ for $j > 0, \mu > 1$ and $C'_{0,1} = \frac{\kappa}{(\rho-1)\kappa+1}$ for $\rho > 1$ are obviously finite at $\kappa = \kappa_0$. Next for $j > 1$ we have $C'_{0,j} = \frac{\kappa}{((j-1)\mu + \rho - 1)\kappa + m(j-1) + 1}$ with denominator $(j-1)(\mu\kappa + m) + (\rho-1)\kappa + 1$ which is nonzero at $\kappa = \kappa_0$ by Lemma 4, since $\rho-1 \leq \mu$. Finally for $1 \leq k < j-1 \leq l$ we have $C'_{k,j} = \kappa((j-k-1)(\mu\kappa + m) + (\mu-1)\kappa + 1)^{-1}$, and the Lemma applies. \square

We restate the result of Theorem 2 applied to the x -monic polynomials ζ_λ^x and $\zeta_{\lambda-\varepsilon(\rho+k\mu)}^x$ (for $\lambda = \Lambda(\mu, s, l, \rho, m)$ and $0 \leq k \leq l$). For $1 \leq i \leq \rho + l\mu$

$$\begin{aligned} \mathcal{D}_i \zeta_\lambda^x &= \sum_{k=0}^l ((l-k)(\kappa\mu + m) + (s+1)(m + \kappa(\mu+1))) u_{i, k+1} \\ &\quad \times \frac{h(\lambda, 1) h(\lambda - \varepsilon(\rho + k\mu), \kappa + 1)}{h(\lambda, \kappa + 1) h(\lambda - \varepsilon(\rho + k\mu), 1)} \zeta_{\lambda - \varepsilon(\rho + k\mu)}^x, \end{aligned}$$

where (the labeling of the points of decrease is now shifted by 1) each $u_{i, k+1} \in RS[1, \rho + l\mu]$ and R is the ring generated by $\{C_{jk} : 1 \leq j < k \leq l+1\} \cup \{C'_{kj} : 0 \leq k < j \leq l+1\} \cup \mathbb{Z}$; also $u_{i, k+1} = 0$ for $k < \frac{i-\rho}{\mu}$. Since $h(\lambda, \kappa + 1) \neq 0$ at $\kappa = \kappa_0$ the polynomial ζ_λ has no poles there. Also the specialization of R is a subring of \mathbb{Q} . For $k = l$ we already have shown that $h(\lambda - \varepsilon(\rho + l\mu), \kappa + 1) \neq 0$ and thus $\zeta_{\lambda - \varepsilon(\rho + l\mu)}^x$ has no poles at κ_0 and the factor $(s+1)(m + \kappa(\mu+1))$ becomes zero. When $0 \leq k < l$ the factor $h(\lambda - \varepsilon(\rho + k\mu), \kappa + 1)$ has a zero at κ_0 . Once we prove that $\zeta_{\lambda - \varepsilon(\rho + k\mu)}^x$ has no pole at κ_0 the proof that $\mathcal{D}_i \zeta_\lambda = 0$ for all i will be complete.

The method of Knop and Sahi [11] was designed to show that the coefficients of the monomials x^β in $h(\lambda, \kappa + 1) \zeta_\lambda^x$ are in $\mathbb{N}_0[\kappa]$, but it is not evident how to use the method to identify the poles when the number of variables is in the range $\ell(\lambda) \leq N < \ell(\lambda) + |\lambda|$. We introduce a different approach.

Definition 7. Let $\alpha, \beta \in \mathbb{N}_0^M$ with $\alpha \triangleright \beta$ and let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$ then say (α, β) is a $(-\frac{m}{n})$ -critical pair if $(n\kappa + m)$ divides $(r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i$ (in $\mathbb{Q}[\kappa]$) for $1 \leq i \leq M$.

In fact the division is in $\mathbb{Z}[\kappa]$ because $\gcd(m, n) = 1$. The definition will be used in the situation $\alpha \in \mathbb{N}_0^{N, P}$ that is, $\ell(\alpha) \leq N$ and $M = \ell(\alpha) + |\alpha|$. See Definitions 2 and 3 for the order \triangleright and the rank function r .

Theorem 3. Suppose $\lambda \in \mathbb{N}_0^{N, P}$ and $\kappa_0 \in \mathbb{Q}, \kappa_0 < 0$; if there does not exist $\beta \in \mathbb{N}_0^N$ such that (λ, β) is a κ_0 -critical pair then κ_0 is not a pole of ζ_λ^x restricted to \mathbb{R}^N .

Proof. Extend the field $\mathbb{Q}(\kappa)$ with N transcendental variables $\{v_1, v_2, \dots, v_N\}$ and let $\mathcal{T} = \sum_{i=1}^N v_i \mathcal{U}_i$. For each $\alpha \in \mathbb{N}_0^N$ the polynomial ζ_α^x is an eigenvector of \mathcal{T} , indeed $\mathcal{T} \zeta_\alpha^x = \sum_{i=1}^N v_i \xi_i(\alpha) \zeta_\alpha^x$. The eigenvalue determines α uniquely for generic κ (with the possible exception of a finite set of negative rationals). Let $C = \{\beta \in \mathbb{N}_0^N : \lambda \triangleright \beta\}$. By the triangularity of the operators $\{\mathcal{U}_i\}$ we have $x^\lambda = \zeta_\lambda^x + \sum_{\beta \in C} B_{\beta\lambda} \zeta_\beta^x$ for certain coefficients $B_{\beta\lambda} \in \mathbb{Q}(\kappa)$. Let

$$\mathcal{T}_\lambda = \prod_{\beta \in C} \frac{\mathcal{T} - \sum_{i=1}^N v_i \xi_i(\beta)}{\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta))},$$

then $\mathcal{T}_\lambda x^\lambda = \zeta_\lambda^x$ (note that the number N of variables is part of the definition of the set C). The numerator of the product is a polynomial in κ, v_1, \dots, v_N (and of course each $\mathcal{D}_i x^\alpha$ is a polynomial with coefficients in $\mathbb{Z}[\kappa]$) thus any (κ) -poles in ζ_λ^x must appear in the set $\{\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta)) : \lambda \triangleright \beta\}$. For any $\beta \in C$ we have $\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta)) = \sum_{i=1}^N v_i ((r(\beta, i) - r(\lambda, i))\kappa + \lambda_i - \beta_i)$. Since any denominator appearing in a coefficient (with respect to the x -monomial basis) of ζ_λ^x must be a factor of $h(\lambda, \kappa + 1)$, all of the terms involving $\{v_i : 1 \leq i \leq N\}$

must cancel out in the calculation of $\mathcal{T}_\lambda x^\lambda$. Thus the irreducible polynomials $\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta))$ must cancel out and the denominators in ζ_λ^x can only arise from reducible terms of the form $\left(\sum_{i=1}^N a_i v_i\right) (\kappa - \kappa_1)$ where $a_1, \dots, a_N, \kappa_1 \in \mathbb{Q}$. This condition is equivalent to (λ, β) being a κ_1 -critical pair. Thus, if there is no κ_0 -critical pair (λ, β) with $\ell(\beta) \leq N$ then κ_0 is not a pole of ζ_λ^x . \square

We will exploit this theorem by directly constructing the unique β such that $(\Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k\mu), \beta)$ is $-\frac{m}{\kappa+1}$ -critical. Here is a numerical example: for $N = 33$ consider $\Lambda(3, 4, 4, 2, 3)$ for the singular value $\kappa_0 = -\frac{3}{4}$ of isotype $(19, 3^4, 2)$, take $k = 1$, then $\lambda = (27^2, 24^2, 23, 21^3, 18^3, 15^3)$ and the unique β such that (λ, β) is $(-\frac{3}{4})$ -critical is $(27^2, 24^2, 2, 0^3, 21^3, 18^3, 3^{22})$. The construction proceeds through several lemmas. Fix k such that $0 \leq k \leq l-1$, let $\lambda = \Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k\mu)$, $L = \ell(\lambda) = \rho + l\mu$; and partition $[1, L]$ into cells $\{I_j : 0 \leq j \leq l\}$, where $I_0 = [1, \rho]$ and $I_j = [\rho + (j-1)\mu + 1, \rho + j\mu]$ for $1 \leq j \leq l$. Then $i \in I_j$ implies $\lambda_i = m(l + s + 1 - j)$, except that $\lambda_{\rho+k\mu} = m(l + s + 1 - k) - 1$. We will show the required β has the values $m(l + s + 1 - j)$ on the cells I_j with $j \leq k$, 0 on I_{k+1} , $m(l + s + 2 - j)$ on I_j with $k < j \leq l$, except $\beta_{\rho+k\mu} = m - 1$, and $\beta_i = m$ for $L + 1 \leq i \leq L + \mu(s + 1) + l + s - k = N + l - k$; also $\ell(\beta) = N + l - k$. Henceforth, suppose that (λ, β) is $-\frac{m}{\kappa+1}$ -critical or $\beta = \lambda$. This holds if and only if the *rank equation*

$$(3.1) \quad r(\beta, i) - i = (\mu + 1) \left(\frac{1}{m} (\lambda_i - \beta_i) \right)$$

is satisfied for all $i \geq 1$. Since $\gcd(m, \mu + 1) = 1$ this implies that $\beta_i \equiv \lambda_i \pmod{m}$ (so with the exception of $\beta_{\rho+k\mu}$ each β_i is divisible by m). Here is a maximum principle for the multiplicity $\#\{j : \beta_j = m\gamma, 1 \leq j \leq L\}$ for any γ . There is a slight difference for the cases $m = 1$ and $m > 1$. The condition $\lambda \supseteq \beta$ implies that any possible values satisfy $\gamma \leq s + l + 1$.

Lemma 5. *Suppose $\gamma \in \mathbb{N}_0$, and $G = \{j : \beta_j = m\gamma, 1 \leq j \leq L\}$, $m > 1$ or $m = 1$ and $\rho + k\mu \notin G$, if G meets two or more cells then $\#G \leq \mu - 1$; additionally, if one of the cells is I_0 then $\#G \leq \rho - 1$.*

Proof. Let G have nonempty intersections with cells $I_{g_1}, I_{g_2}, \dots, I_{g_u}$ with $0 \leq g_1 < g_2 < \dots < g_u \leq l$. By hypothesis $\rho + k\mu \notin G$ (if $m > 1$ then m does not divide $\beta_{\rho+k\mu}$) and so $i \in G \cap I_{g_a}$ implies $\lambda_i = m(s + l + 1 - g_a)$. Each $G \cap I_{g_a}$ is an interval $[i_a, j_a]$; indeed suppose $i, j \in G \cap I_{g_a}$ and $i < j$, then by equation (3.1) $r(\beta, i) - i = (\mu + 1) \left(\frac{1}{m} (m(s + l + 1 - g_a) - m\gamma) \right) = r(\beta, j) - j$; thus $r(\beta, j) = r(\beta, i) + j - i$. Since $\beta_j = \beta_i$ this implies that $\beta_b = \beta_i = m\gamma$ for $i \leq b \leq j$. For $0 \leq a < u$ we have that $r(\beta, i_{a+1}) = r(\beta, j_a) + 1$ and we combine the two equations

$$\begin{aligned} r(\beta, i_{a+1}) - i_{a+1} &= (\mu + 1) (l + s + 1 - g_{a+1} - \gamma), \\ r(\beta, j_a) - j_a &= (\mu + 1) (l + s + 1 - g_a - \gamma) \end{aligned}$$

to obtain $i_{a+1} = j_a + 1 + (\mu + 1)(g_{a+1} - g_a)$. Then $\#G = \sum_{a=1}^u (j_a - i_a + 1) = u + j_u - i_1 - \sum_{a=1}^{u-1} (i_{a+1} - j_a) = u + j_u - i_1 - \sum_{a=1}^{u-1} (1 + (\mu + 1)(g_{a+1} - g_a)) = 1 + j_u - i_1 - (\mu + 1)(g_u - g_1)$. But $j_u \leq \rho + g_u\mu$ and $i_1 \geq \rho + (g_1 - 1)\mu + 1$ for $g_1 \geq 1$ while $i_1 \geq 1$ for $g_1 = 0$. This shows that $\#G \leq \mu - (g_u - g_1)$ for $g_1 \geq 1$ and $\#G \leq \rho - g_u$ if $g_1 = 0$. In both cases $\#G \leq \mu - 1$. \square

Lemma 6. *Suppose $m = 1, \gamma \in \mathbb{N}_0, G = \{j : \beta_j = \gamma, 1 \leq j \leq L\}$, and $G \setminus \{\rho + k\mu\}$ meets two or more cells, if $\rho + k\mu = \min(G)$ and $G \neq [\rho + k\mu, \rho + (k+1)\mu]$ then $\#G \leq \mu$, otherwise ($\rho + k\mu \neq \min(G)$) then $\#G \leq \mu - 1$.*

Proof. By hypothesis $G \neq [\rho + k\mu, \rho + (k+1)\mu]$. We can apply the previous argument if we replace I_k by $I_k \setminus \{\rho + k\mu\}$ and I_{k+1} by $I_{k+1} \cup \{\rho + k\mu\} = [\rho + k\mu, \rho + (k+1)\mu]$. If $g_1 \neq k+1$ then as before $\#G \leq \mu - (g_u - g_1) \leq \mu - 1$. If $g_1 = k+1$ and $i_1 \geq \rho + k\mu + 1$ the same conclusion results. When $g_1 = k+1$ and $i_1 = \rho + k\mu$, that is, $\min G = \rho + k\mu$, the calculation yields the bound $\#G \leq \mu + 1 - (g_u - k - 1) \leq \mu$. \square

The two lemmas show that $\#\{j : \beta_j = \gamma m, 1 \leq j \leq L\} \leq \mu$ for any $\gamma \in \mathbb{N}_0$, except when $m = 1$ and $\{j : \beta_j = \gamma\} = [\rho + k\mu, \rho + (k+1)\mu]$ of cardinality $\mu + 1$. Next we show $\beta_{L+1} \leq m$.

Lemma 7. *Either $\beta_{L+1} = m$ and $r(\beta, L+1) = L - \mu$, or $\beta_{L+1} = 0$ and $r(\beta, L+1) = L + 1, \ell(\beta) = L$.*

Proof. Denote $\frac{\beta_{L+1}}{m}$ by b ; note that $b \in \mathbb{N}_0$. First we show $b \leq l$: by equation 3.1 $r(\beta, L+1) = L + 1 - (\mu + 1)b \geq 1$ and so $b \leq \frac{L+1}{\mu+1} \leq (l+1) \frac{\mu}{\mu+1} < l+1$. Let $a_0 = \#\{j : 1 \leq j \leq L, \beta_j < \beta_{L+1}\}$ and $a_1 = \#\{j : j > L, \beta_j > \beta_{L+1}\}$ then $r(\beta, L+1) = L + 1 - a_0 + a_1 \geq L + 1 - a_0$. We claim $a_0 \leq b\mu + 1$. If $m > 1$ then $a_0 = \sum_{i=0}^{b-1} \#\{j : \beta_j = im, 1 \leq j \leq L\} + \#\{j : \beta_j = cm - 1, c \leq b\}$. By the maximum principle $a_0 \leq b\mu + 1$. If $m = 1$ then $a_0 = \sum_{i=0}^{b-1} \#\{j : \beta_j = i, 1 \leq j \leq L\}$; at most one of these sets can have cardinality $\mu + 1$ and again $a_0 \leq b\mu + 1$. Then $L + 1 - (\mu + 1)b = r(\beta, L+1) \geq L - b\mu$, that is, $b \leq 1$. If $b = 1$ then $r(\beta, L+1) = L - \mu$. If $b = 0$ then $r(\beta, L+1) = L + 1$ which implies $\beta_j = 0$ for all $j > L$. The hypothesis $\lambda \succeq \beta$ implies $L = \ell(\lambda) \leq \ell(\beta)$. \square

In fact, $\beta_{L+1} = 0$ implies $\beta = \lambda$ and $\beta_{L+1} = m$ corresponds to a unique solution β with $\ell(\beta) = N + l - k$.

Lemma 8. *Suppose that $\beta_{L+1} = m$ then $\beta_{\rho+k\mu} = m - 1, \beta_i = 0$ for $i \in I_{k+1}, \beta_i = m$ for $L + 1 \leq i \leq N + l - k$ and $\ell(\beta) = N + l - k$.*

Proof. Let $a_0 = \#\{j : 1 \leq j \leq L, \beta_j \geq m\}$, $a_1 = \#\{j : L < j, \beta_j > m\}$, $G_0 = \{j : 1 \leq j \leq L, \beta_j = 0\}$, $G_1 = \{j : \beta_j = m - 1 > 0\}$, and $a_2 = \#G_0 + \#G_1$ where G_1 is empty when $m = 1$. Then $L - \mu = r(\beta, L+1) = a_0 + a_1 + 1$ and $L = a_0 + a_2$, thus $a_2 = \mu + 1 + a_1 \geq \mu + 1$. But by the maximum principle $a_2 \leq \mu + 1$, hence $a_1 = 0$ and $a_2 = \mu + 1$. If $m > 1$ then $\#G_0 \leq \mu$ implying that $\#G_1 = 1$ and $G_1 = \{\rho + k\mu\}$, also $\#G_0 = \mu$ and thus $G_0 = I_j$ for some $j \neq 0, k$ by Lemma 5. If $m = 1$ then $G_0 = [\rho + k\mu, \rho + (k+1)\mu]$ by Lemma 6. Let $r_0 = r(\beta, \rho + k\mu)$, then $r(\beta, \rho + (j-1)\mu + 1) = r_0 + 1$; if $m = 1$ then $\beta_{\rho+k\mu} = \beta_{\rho+k\mu+1} = 0$ and $j = k+1$, while for $m > 1$ we have $\beta_{\rho+k\mu} = m - 1$ which is the unique minimum of $\{\beta_j : 1 \leq j \leq L, \beta_j > 0\}$ and $\beta_{\rho+(j-1)\mu+1}$ is the first occurrence of 0. By equation 3.1

$$\begin{aligned} r_0 - (\rho + k\mu) &= \frac{\mu + 1}{m} (m(s + l + 1 - k) - 1 - (m - 1)) \\ &= (\mu + 1)(s + l - k), \\ r_0 + 1 - (\rho + (j - 1)\mu + 1) &= (\mu + 1)(s + l + 1 - j). \end{aligned}$$

Thus $r_0 = \rho + k\mu + (\mu + 1)(s + l - k)$ and $(j - 1 - k)\mu = (\mu + 1)(j - 1 - k)$, that is $j = k + 1$. But $r_0 = \#\{j : \beta_j \geq m\} + 1 = L - \mu + \#\{j : L < j, \beta_j = m\}$ and so $\#\{j : L < j, \beta_j = m\} = \mu s + \mu + s + l - k = N - L + l - k$. This shows $\ell(\beta) = N + l - k \geq N + 1$. \square

Certainly this, together with a proof that $\ell(\beta) = L$ implies $\beta = \lambda$, is enough for the main purpose, but with not much more work we can show that β is unique. In fact we will show that $\beta_{L+1} = m$ implies that for $i \in I_j$ $\beta_i = m(s + l + 1 - j)$ for $j \leq k$ and $\beta_i = m(s + l + 2 - j)$ for $j > k + 1$, except $\beta_{\rho+k\mu} = m - 1$.

Lemma 9. *Suppose that $\beta_{L+1} = m$, then $\beta_i = \lambda_i$ for all $i < \rho + k\mu$ and $\beta_i = \lambda_i + m$ for $\rho + (k + 1)\mu + 1 \leq i \leq \ell(\beta)$.*

Proof. For $0 \leq i \leq s + l$ let $M_i = \#\{j : 1 \leq j \leq L, \beta_j = m(s + l + 1 - i)\}$. Since $\lambda_i = m(s + l + 1)$ for $1 \leq i \leq \rho$ when $k \geq 1$, and for $1 \leq i \leq \rho - 1$ when $k = 0$, the condition $\lambda \triangleright \beta$ (thus $\lambda \succeq \beta^+$) implies $M_0 \leq \rho$ or $\rho - 1$ respectively. The maximum principle (Lemmas 5 and 6) implies that $M_i \leq \mu$ for $1 \leq i < l + 1$ (from the previous lemma in which $\beta_{\rho+k\mu} = m - 1$ was determined). Further $\sum_{i=0}^{s+l} M_i = L - \mu - 1 = \rho + (l - 1)\mu - 1$, that is, $\sum_{i=l}^{s+l} M_i = \rho - M_0 + \sum_{i=1}^{l-1} (\mu - M_i) - 1$. Also

$$\begin{aligned} |\beta| &= \sum_{i=0}^{s+l} M_i m(s + l + 1 - i) + m - 1 + m(\mu s + \mu + s + l - k) \\ &= \rho m(s + l + 1) + m\mu \sum_{i=1}^l (s + i) - 1, \end{aligned}$$

and so

$$\begin{aligned} \sum_{i=l}^{s+l} M_i (s + l + 1 - i) &= (\rho - M_0)(s + l + 1) \\ &\quad + \sum_{i=1}^{l-1} (\mu - M_i)(s + l + 1 - i) - (s + l + 1 - k). \end{aligned}$$

Let j be defined by $M_0 = \rho$, $M_i = \mu$ for $1 \leq i \leq j$ and $M_{j+1} \leq \mu - 1$, that is, $j \geq 0$, while $j = -1$ when $M_0 \leq \rho - 1$. The hypothesis $\lambda \succeq \beta^+$ implies $j < k$ (or else $\sum_{i=1}^{\rho+k\mu} \beta_i^+ > \sum_{i=1}^{\rho+k\mu} \lambda_i$). Then for $j = -1$ we have

$$\begin{aligned} \sum_{i=l}^{s+l} M_i (s + l + 1 - i) &= (\rho - 1 - M_0)(s + l + 1) + \sum_{i=1}^{l-1} (\mu - M_i)(s + l + 1 - i) + k \\ &\geq (s + 2) \left(\rho - 1 - M_0 + \sum_{i=1}^{l-1} (\mu - M_i) \right) = \sum_{i=l}^{s+l} M_i (s + 2), \end{aligned}$$

and for $j \geq 0$

$$\begin{aligned} \sum_{i=l}^{s+l} M_i (s + l + 1 - i) &= (\rho - M_0)(s + l + 1) + \sum_{i=1, i \neq j+1}^{l-1} (\mu - M_i)(s + l + 1 - i) \\ &\quad + (\mu - 1 - M_{j+1})(s + l - j) + (k - 1 - j) \\ &\geq \sum_{i=l}^{s+l} M_i (s + 2), \end{aligned}$$

since each coefficient is nonnegative, but then $M_i = 0$ for all $i \geq l$. The nonnegativity of each term on the right hand sides implies $j = k - 1$ and if $k = 0$ then $M_0 = \rho - 1$ and $M_i = \mu$ for $1 \leq i \leq l - 1$, or else $M_0 = \rho, M_k = \mu - 1$ and $M_i = \mu$ for $1 \leq i \leq l - 1, i \neq k$. Let $G_i = \{j : 1 \leq j \leq L, \beta_j = m(s + l + 1 - i)\}$ for $2 \leq i \leq l + 1$. By Lemma 5 for each i satisfying $1 \leq i \leq l - 1, i \neq k$ there is u_i such that $G_i = I_{u_i}$ and $u_i \neq k, k + 1$.

If $k = 0$ then $G_0 = [1, \rho - 1]$ since all other cells are of cardinality μ . For each G_i with $i \geq 1$ the rank of the first coordinate is $\rho + (i - 1)\mu$, that is $r(\beta, \rho + (u_i - 1)\mu + 1) = \rho + (i - 1)\mu$. Then by equation 3.1

$$\rho + (i - 1)\mu - (\rho + (u_i - 1)\mu + 1) = (\mu + 1)((s + l + 1 - u_i) - (s + l + 1 - i)),$$

thus $u_i = i + 1$, for $1 \leq i \leq l - 1$. If $k > 0$ we have shown $\#G_0 = \rho$ and $\#G_k = \mu - 1$. If $\rho \leq \mu - 1$ then neither G_0 nor G_k can meet I_0 and another cell, by Lemma 5. If additionally $\rho < \mu - 1$ then $G_0 = I_0$ and $G_k = [\rho + (k - 1)\mu + 1, \rho + k\mu - 1]$. If $\rho = \mu - 1$ then it is not possible for $G_k = I_0$ because then $r(\beta, 1) = \rho + (k - 1)\mu + 1$ and equation 3.1 yields $\rho + (k - 1)\mu = (\mu + 1)k$, that is $k + 1 = 0$. As before, $G_0 = I_0$ and $G_k = I_k \setminus \{\rho + k\mu\}$. If $\rho = \mu$ then $G_k = I_k \setminus \{\rho + k\mu\}$ and $G_0 = I_{u_0}$ for some u_0 . The needed ranks for β are $r(\beta, \rho + (u_i - 1)\mu + 1) = \rho + (i - 1)\mu + 1$ if $i < k$ and $= \rho + (i - 1)\mu$ if $k < i \leq l - 1$. Similarly to the case $k = 0$ this implies that $u_i = i$ for $i < k$ and $u_i = i + 1$ for $k < i \leq l - 1$. \square

It remains to show that $\ell(\beta) \leq L$ implies $\beta = \lambda$.

Lemma 10. *Suppose $\beta_{L+1} = 0$, then $\beta = \lambda$.*

Proof. The hypothesis implies $\beta_i = 0$ for all $i > L$ (the rank equation showed $r(\beta, L + 1) = L + 1$ thus $i > L + 1$ implies $\beta_i = 0$). The condition $\lambda \succeq \beta^+$ implies $\beta_L^+ \geq \lambda_L = m(s + 1)$ (since $|\lambda| - \lambda_L \geq |\beta| - \beta_L^+$). For $1 \leq i \leq l + 1$ let $G_i = \{j : \beta_j = m(s + l + 1 - i)\}$ and $M_i = \#G_i$. Firstly let $m > 1$, then $\beta_{\rho+k\mu} = m(s + l + 1 - j_0) - 1$ for some j_0 in $0 \leq j \leq l - 1$, thus $\rho + l\mu = \sum_{i=0}^l M_i + 1$ and $\rho - M_0 + \sum_{i=1}^l (\mu - M_i) = 1$. Also $M_0 \leq \rho$ because $\lambda \succeq \beta^+$ and $M_i \leq \mu$ for $1 \leq i \leq l$ by Lemma 5. Hence either $M_0 = \rho - 1$ and $M_i = \mu$ for $1 \leq i \leq l$, or for some $j > 0$ $M_j = \mu - 1$ and $M_0 = \rho, M_i = \mu$ for $1 \leq i \leq l, i \neq j$. Now

$$\begin{aligned} |\beta| &= \rho m(s + l + 1) + \mu m \sum_{i=1}^l (s + l + 1 - i) - 1 \\ &= M_0 m(s + l + 1) + m \sum_{i=1}^l M_i (s + l + 1 - i) + m(s + l + 1 - j_0) - 1, \end{aligned}$$

and substituting the known values for M_i we obtain $j_0 = 0$ if $M_0 = \rho - 1$ else $j_0 = j$. Then $r(\beta, \rho + k\mu) = \rho + j_0\mu$ and the rank equation at $\rho + k\mu$ yields $(j_0 - k)\mu = (\mu + 1)(j_0 - k)$ and so $j_0 = k$, that is, $\beta_{\rho+k\mu} = \lambda_{\rho+k\mu}$. Similarly to the previous lemma let $G_i = \{j : \beta_j = m(s + l + 1 - i)\} = I_{u_i}$ for $0 \leq i \leq l, i \neq k$ and some $u_i \neq k$, treating the special cases $\rho < \mu - 1, \rho = \mu - 1$ and $\rho = \mu$ as before. Again $r(\beta, \rho + (i - 1)\mu + 1) = \rho + (u_i - 1)\mu + 1$ and the rank equation shows $u_i = i$. Also $G_k = I_k \setminus \{\rho + k\mu\}$. Thus $\beta = \lambda$.

Secondly let $m = 1$. Then $\lambda_i = s + l - k$ for $\rho + k\mu \leq i \leq \rho + (k + 1)\mu$. Also $\rho \geq M_0$ because $\lambda \succeq \beta^+$. There are two equations involving M_i :

$$(3.2) \quad (\rho - M_0) + \sum_{i=1}^l (\mu - M_i) = 0,$$

$$(3.3) \quad (\rho - M_0)(s + l + 1) + \sum_{i=1}^l (\mu - M_i)(s + l + 1 - i) = 1.$$

Equation 3.3 shows that $\rho \geq M_0$ and $\mu \geq M_i$ for all i is impossible, hence there is at least one value, say M_j , such that $M_j > \mu$. By the maximum principle $M_j = \mu + 1$ and $M_i \leq \mu$ for all $i \neq j$. Substituting these conditions in equation 3.2 shows that for some j_0 , $M_{j_0} = \mu - 1$ ($M_0 = \rho - 1$ if $j_0 = 0$) and $M_i = \mu$ for all $i \neq j_0, j$, and $M_0 = \rho$ unless $j_0 = 0$. Substitute these values in equation 3.3 to obtain $j_0 = j - 1$. By Lemma 6 $G_j = [\rho + k\mu, \rho + (k + 1)\mu]$, also $r(\beta, \rho + k\mu) = \rho + (j - 1)\mu$. Then the rank equation shows $(j - 1 - k)\mu = (\mu + 1)((s + l - k) - (s + l + 1 - j)) = (\mu + 1)(j - 1 - k)$ and thus $j = k + 1$. Similarly to the previous arguments, for each $i \neq k, k + 1$ there exist u_i such that $G_i = I_{u_i}$. Since $M_k + M_{k+1} = 2\mu$ (or $\rho + \mu$ if $k = 0$) we have $r(\beta, \rho + (u_i - 1)\mu + 1) = \rho + (i - 1)\mu + 1$ and the rank equation shows $u_i = i$. This accounts for all of $[1, L]$ except for $[1, \rho]$ and $[\rho + (k - 1)\mu + 1, \rho + k\mu - 1]$. There are several cases for ρ : if $k = 0$ then $G_0 = [1, \rho - 1]$ by elimination; if $k \geq 1$ and $\rho = \mu$ then $G_0 = I_{u_0}$ and the rank equation shows $u_0 = 0$, and $G_k = [\rho + (k - 1)\mu + 1, \rho + k\mu - 1]$; if $k \geq 1$ and $\rho = \mu - 1$ then by Lemma 6 G_0 can not meet both I_0 and I_k thus either $G_0 = I_0$ or $G_0 = I_k \setminus \{\rho + k\mu\}$ and the rank equation implies the latter can not happen; if $k \geq 1$ and $\rho < \mu - 1$ then by the same Lemma $G_k = I_k \setminus \{\rho + k\mu\}$, forcing $G_0 = I_0$. Thus $\beta = \lambda$. \square

The lemmas together provide the proofs of the following theorems.

Theorem 4. *Let $\lambda = \Lambda(\mu, s, l, \rho, m)$ and $\lambda^{(k)} = \lambda - \varepsilon(\rho + k\mu)$ for $0 \leq k \leq l - 1$. Then there exists a unique β so that $(\lambda^{(k)}, \beta)$ is $\left(-\frac{m}{\mu+1}\right)$ -critical and $\ell(\beta) = N + l - k > N$, where $N = (s + l + 1)\mu + s + \rho$.*

Theorem 5. *Let $\lambda = \Lambda(\mu, s, l, \rho, m)$ and $N = (s + l + 1)\mu + s + \rho$ then ζ_λ^x is a singular polynomial for S_N with singular value $-\frac{m}{\mu+1}$.*

In the next section we study the irreducible representation associated to ζ_λ^x , in particular, an explicit basis for the span of its S_N -orbit.

4. ASSOCIATED S_N -MODULES

Using Murphy's construction [13] of Young's seminormal representations we can give a complete description of the S_N -orbit of ζ_λ^x . From the formula (valid for all κ and for all polynomials f)

$$\sum_{i=1}^N x_i \mathcal{D}_i f(x) = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{1 \leq i < j \leq N} (f(x) - f(x(i, j)))$$

we note that a homogeneous singular polynomial f must satisfy $(\deg f) f = -\kappa \omega f$ where $\omega = \sum_{1 \leq i < j \leq N} (1 - (i, j))$. But ω is in the center of $\mathbb{Z}S_N$ and the eigenvalues for any isotype are known (Young's formula). Indeed for any node (i, j) in the

Ferrers diagram of a partition τ (with $|\tau| = N$), the content is defined to be $c((i, j)) = j - i$, then $\omega f = \left(\binom{N}{2} - \sum_{(i,j) \in \tau} c((i, j)) \right) f$ whenever f is of isotype τ . Denote the eigenvalue by $\tau(\omega)$, then $\tau(\omega) = \binom{N}{2} - \frac{1}{2} \sum_{i=1}^{\ell(\tau)} \tau_i (\tau_i + 1 - 2i)$. As a function on partitions the eigenvalue is strictly decreasing with respect to the dominance order.

Lemma 11. *Suppose $\sigma, \tau \in \mathbb{N}_0^{N,P}$, $|\sigma| = |\tau|$ and $\sigma \prec \tau$ then $\sum_{(i,j) \in \sigma} c((i, j)) < \sum_{(i,j) \in \tau} c((i, j))$, and $\sigma(\omega) > \tau(\omega)$.*

Proof. By the theorems (1.15) and (1.16) in Macdonald [12, p.9] it suffices to prove the inequality for $\tau = \sigma + \varepsilon(i) - \varepsilon(j)$ with $i < j$ (this is a “raising operator”). Then $\sum_{(i,j) \in \tau} c((i, j)) = \sum_{(i,j) \in \sigma} c((i, j)) + (\sigma_i - \sigma_j) + (j + 1 - i)$. \square

Recall the singular polynomials ζ_λ^x associated to two-part partitions $\tau = (\mu, N - \mu)$ with $\lambda = (m^{N-\mu})$ and $\gcd(m, \mu + 1) < \frac{\mu+1}{N-\mu}$; then $\tau(\omega) = (\mu + 1)(N - \mu)$ and $\deg \zeta_\lambda^x = m(N - \mu)$. For $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$ we find

$$\begin{aligned} \tau(\omega) &= (\mu + 1) \left(\rho(s + l + 1) + \frac{1}{2} \mu l (l + 2s + 1) \right) \\ &= \frac{\mu + 1}{m} |\Lambda(\mu, s, l, \rho, m)|. \end{aligned}$$

Theorem 6. *For $\lambda = \Lambda(\mu, s, l, \rho, m)$ and $\kappa = -\frac{m}{\mu+1}$ the singular polynomial ζ_λ^x on \mathbb{R}^N is of isotype $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$ ($|\tau| = N$).*

Proof. For any ζ_σ^x with $\sigma \in \mathbb{N}_0^N$ if $\sigma_i = \sigma_{i+1}$ for some i then $(i, i + 1) \zeta_\sigma^x = \zeta_\sigma^x$. Thus ζ_λ^x is invariant under $S_{[1, \rho]} \times \prod_{j=1}^l S_{[\rho+(j-1)\mu+1, \rho+j\mu]} \times S_{[\rho+l\mu+1, N]}$, and this group is conjugate to S_τ (the direct product $\prod_i S_{\tau_i}$). Thus $E = \text{span}_{\mathbb{Q}} \{w \zeta_\lambda^x : w \in S_N\}$ is isomorphic to a submodule of the representation of S_N induced up from 1_{S_τ} , the identity representation of S_τ . By a classical theorem (see Macdonald [12, p.115]) this decomposes as a direct sum with one component of isotype τ and all other components of isotypes σ with $\sigma \succ \tau$. Any $f \in E$ is singular and f can not be of isotype $\sigma \succ \tau$ because $\deg f = |\lambda| = \frac{m}{\mu+1} \tau(\omega) > \frac{m}{\mu+1} \sigma(\omega)$ by the Lemma. \square

The same method proves the following.

Proposition 8. *For $\frac{N}{2} \leq \mu < N$, $\gcd(m, \mu + 1) < \frac{\mu+1}{N-\mu}$, and $\lambda = (m^{N-\mu})$ the singular polynomial ζ_λ^x on \mathbb{R}^N for $\kappa = -\frac{m}{\mu+1}$ is of isotype $(\mu, N - \mu)$.*

We turn to the application of Murphy’s results. For any given isotype he determined the eigenvalues and eigenvectors of the commuting operators

$\left\{ \sum_{j=1}^{i-1} (i, j) : 2 \leq i \leq N \right\}$ (Jucys-Murphy elements). However the results have to be read in reverse in a certain sense.

Proposition 9. *Suppose f is a singular polynomial for $\kappa = \kappa_0 \in \mathbb{Q}$ and $1 \leq i \leq N$, then $\mathcal{U}_i f = f + \kappa_0 \sum_{j=i+1}^N (i, j) f$.*

Proof. We have the commutation $\mathcal{D}_i(x_i f) = x_i \mathcal{D}_i f + f + \kappa \sum_{j \neq i} (i, j) f$. Now set $\kappa = \kappa_0$ and note that $\mathcal{U}_i f = \mathcal{D}_i(x_i f) - \kappa \sum_{j < i} (i, j) f$. \square

Denote the Murphy elements $\omega_i = \sum_{j=N-i+2}^N (N+1-i, j)$ for $2 \leq i \leq N$ and let $\omega_1 = 0$ (as a transformation); then $\mathcal{U}_i f = f + \kappa_0 \omega_{N+1-i} f$ for singular polynomials. Suppose that ζ_α^x is singular for $\kappa = \kappa_0$ and $\alpha = w\lambda$, some $w \in S_N$ (recalling that $\mathcal{U}_i \zeta_\alpha^x = ((N-r(\alpha, i))\kappa + \alpha_i + 1) \zeta_\alpha^x$), then $\omega_{N+1-i} \zeta_\alpha^x = \left(N - r(\alpha, i) + \frac{\alpha_i}{\kappa_0}\right) \zeta_\alpha^x$. A *standard Young tableau* (SYT) of shape τ is a one-to-one assignment of the numbers $\{1, \dots, N\}$ to the nodes of the Ferrers diagram so that the entries increase in each row and in each column. Let $\eta_i(T)$ be the content of the node containing the value i , $1 \leq i \leq N$. Murphy constructed a basis $\{f_T : T \text{ is an SYT of shape } \tau\}$ for the irreducible representation of isotype τ and $\omega_i f_T = \eta_i(T) f_T$ for each i and T . There is an order on SYT's of given shape (for details see [13, p.288]) and the maximum SYT in this order, denoted by T_0 , is produced by entering the numbers $1, 2, \dots, N$ row by row (the first row is $1, \dots, \tau_1$, the second is $\tau_1 + 1, \dots, \tau_1 + \tau_2$ and so forth).

Definition 8. Suppose T is an SYT of shape τ , with $\tau \in \mathbb{N}_0^{N,P}$ and $|\tau| = N$, then let $rw(i, T)$, $cm(i, T)$ denote the row and column respectively of the node of T containing i , for $1 \leq i \leq N$. Let $t(i, \tau)$ (or $t(i)$) = $(rw(i, T_0), cm(i, T_0))$, considered as a labeling of the nodes in the diagram of τ .

In this notation $\eta_i(T) = cm(i, T) - rw(i, T)$.

Proposition 10. Let $\lambda = \Lambda(\mu, s, l, \rho, m)$ (hypotheses as in Definition 6) then $N - k + \frac{\lambda_k}{\kappa_0} = c(t(N+1-k)) = \eta_{N+1-k}(T_0)$ for $1 \leq k \leq N$

Proof. Let $c_k = N - k + \frac{\lambda_k}{\kappa_0} = N - i - (\mu + 1) \frac{\lambda_k}{m}$ for $1 \leq k \leq N$, then for $k = \rho + 1 - j$ and $1 \leq j \leq \rho$ we have $c_k = j - (l + 2)$, for $k = \rho + (l + 1 - i)\mu + 1 - j$ with $1 \leq i \leq l$ and $1 \leq j \leq \mu$ we have $c_k = j - (i + 1)$, and finally for $k = N + 1 - j$ with $1 \leq j \leq N - l(\lambda) = s(\mu + 1) + \mu = \tau_1$ we have $c_k = j - 1$. Thus $c_k = c(t(N+1-k))$. \square

For a partition $\lambda \in \mathbb{N}_0^{N,P}$ say that $w \in S_N$ is λ -rank-preserving if $\lambda_i = \lambda_{i+1}$ implies $w(i) < w(i+1)$ for $1 \leq i < N$. In general $(w\lambda)_{w(i)} = \lambda_i$, so this property implies $r(w\lambda, i) = r(\lambda, w^{-1}(i)) = w^{-1}(i)$ for $1 \leq i \leq N$ and

$$\mathcal{U}_i \zeta_{w\lambda}^x = \left((N - w^{-1}(i))\kappa + \lambda_{w^{-1}(i)} + 1\right) \zeta_{w\lambda}^x$$

for generic κ . In particular, if $\lambda = \Lambda(\mu, s, l, \rho, m)$, w is λ -rank-preserving and $\zeta_{w\lambda}^x$ is singular (for $\kappa = \kappa_0$) then

$$\omega_{N+1-i} \zeta_{w\lambda}^x = c(t(N+1-w^{-1}(i))) \zeta_{w\lambda}^x.$$

Let w_0 be the ‘‘reversing’’ (longest) element of S_N , that is $w_0(i) = N + 1 - i$ for $1 \leq i \leq N$ (note $w_0^{-1} = w_0$). Thus $\omega_i \zeta_{w\lambda}^x = c(t(w_0 w w_0(i))) \zeta_{w\lambda}^x$. On the other hand suppose $u \in S_N$ and the action of u on T_0 produces an SYT denoted by T (u acts on the entries of T_0), then the node $t(i)$ contains $u(i)$. Thus $\omega_i f_T = \eta_i(T) f_T = c(t(u^{-1}(i))) f_T$ and f_T has the same respective eigenvalues for $\{\omega_i : 1 \leq i \leq N\}$ as $\zeta_{w\lambda}^x$ for $w = w_0^{-1} u w_0$ provided that w is λ -rank-preserving. But this is a consequence of T being an SYT ($\lambda_i = \lambda_{i+1}$ implies $N - i$ and $N + 1 - i$ are in the same row of T_0 thus $u(N - i)$ and $u(N - i + 1)$ are in the same row of T , with $u(N - i) < u(N - i + 1)$, that is, $w(i) < w(i + 1)$). Further it is easy to

describe $w\lambda$ corresponding to a given SYT T : let $\gamma_1 = 0$ and $\gamma_j = m(s + j - 1)$ for $2 \leq j \leq l+2$, then for any i (with $1 \leq i \leq N$) let $j = rw(i, T)$ and $(w\lambda)_{N+1-i} = \gamma_j$. This shows that the possible $w\lambda$ corresponding to SYT's are exactly the reverse lattice permutations of λ . A *reverse lattice permutation* $w\lambda$ of λ is defined by the property that every right substring $(w\lambda)_{N+1-j} (w\lambda)_{N+2-j} \dots (w\lambda)_N$ (for $1 \leq j \leq N$) has at least as many entries of γ_i as of γ_{i+1} for each i . The set of corresponding $w \in S_N$ serves as an index set, namely $E_\tau = \{w \in S_N : w_0 w w_0 T_0 \text{ is an SYT}\}$.

We show that $\{\zeta_{w\lambda}^x : w \in E_\tau\}$ is a basis for the S_N -module (isotype τ) generated by ζ_λ^x .

Theorem 7. *Let $w \in E_\tau$ then $\zeta_{w\lambda}^x$ is singular (for $\kappa = \kappa_0$ on \mathbb{R}^N) and $\{\zeta_{w\lambda}^x : w \in E_\tau\}$ is a basis for $\text{span}_{\mathbb{Q}}\{w\zeta_\lambda^x : w \in S_N\}$, on which S_N acts by Young's seminormal representation, where $\zeta_{w\lambda}^x$ corresponds to f_T with $T = w_0 w w_0 T_0$.*

Proof. By Proposition 1 if $\zeta_{w\lambda}^x$ has no pole at κ_0 , for some $w \in S_N$, and $a = \kappa(\kappa(r(w\lambda, i+1) - r(w\lambda, i)) + \lambda_{w^{-1}(i)} - \lambda_{w^{-1}(i+1)})^{-1}$ does not evaluate to ± 1 at $\kappa = \kappa_0$, for some i with $\lambda_{w^{-1}(i)} > \lambda_{w^{-1}(i+1)}$ then $\zeta_{(i, i+1)w\lambda}^x$ does not have a pole at κ_0 (the formula is $(i, i+1)\zeta_{w\lambda}^x = a\zeta_{w\lambda}^x + (1 - a^2)\zeta_{(i, i+1)w\lambda}$). By Proposition 10 $a = (c(t(N+1 - w^{-1}(i))) - c(t(N+1 - w^{-1}(i+1))))^{-1}$ at $\kappa = \kappa_0$. Each SYT T of shape τ is the result of a (finite) sequence $\{(i_j, i_j + 1) : 1 \leq j \leq n\}$ of adjacent transpositions applied to T_0 , such that if $T_j = (i_j, i_j + 1)T_{j-1}$ then $rw(i_j, T_{j-1}) < rw(i_j + 1, T_{j-1})$. This also implies that T_j is lower in the order on tableaux as used in [13].

For any SYT T there are four possibilities for the locations of $i, i+1$ and Murphy [13, p.292] derived the expansion of $(i, i+1)f_T$ in each case: if $rw(i, T) = rw(i+1, T)$ then $(i, i+1)f_T = f_T$, if $cm(i, T) = cm(i+1, T)$ then $(i, i+1)f_T = -f_T$ and if $rw(i, T) < rw(i+1, T)$ then $(i, i+1)f_T = af_T + (1 - a^2)f_{(i, i+1)T}$ where $a = (\eta_i(T) - \eta_{i+1}(T))^{-1}$ (the fourth case, $rw(i, T) > rw(i+1, T)$ follows from the previous by interchanging T and $(i, i+1)T$; also $rw(i, T) < rw(i+1, T)$ implies $cm(i, T) > cm(i+1, T)$ thus $0 < a \leq \frac{1}{2}$). As remarked before, if $w\lambda$ corresponds to an SYT T with $rw(i, T) < rw(i+1, T)$ then $(w\lambda)_{N+1-i} < (w\lambda)_{N-i}$. Let $\beta = (N - i, N - i + 1)w\lambda$, then $\zeta_\beta^x = (1 - a^2)^{-1}((N - i, N - i + 1)\zeta_{w\lambda}^x - a\zeta_{w\lambda}^x)$ with the same a that appears in the expression for $f_{(i, i+1)T}$ in terms of f_T (note $w_0(i, i+1)w_0 = (N - i, N - i + 1)$). Since f_{T_0} has the same eigenvalues for $\{\omega_i\}$ as ζ_λ^x this argument used inductively (on the number of adjacent transpositions linking T_0 to T) shows that $\{\zeta_{w\lambda}^x : w \in E_\tau\}$ transforms according to the seminormal representation, for the isotype τ . Again suppose $\zeta_{w\lambda}^x$ corresponds to the SYT T (that is $T = w_0 w w_0 T_0$); if $rw(i, T) = rw(i+1, T)$ then $(w\lambda)_{N+1-i} = (w\lambda)_{N-i}$ and $\zeta_{w\lambda}^x$ is invariant under $(N - i, N - i + 1)$, while if $cm(i, T) = cm(i+1, T)$ then $\eta_i(T) - \eta_{i+1}(T) = 1$ and the equation $(N - i, N - i + 1)\zeta_{w\lambda}^x = -\zeta_{w\lambda}^x$ is a consequence of the fact that S_N acts on the basis $\{\zeta_{w\lambda}^x : w \in E_\tau\}$ just as on $\{f_T\}$. \square

The concept of reverse lattice permutations of λ provides a concise labeling of the singular polynomials of isotype τ .

5. CONCLUSION

Here is a description of how to find the isotype τ and label λ for the singular value $\kappa = -\frac{m}{n}$, given a pair (m, n) with $2 \leq n \leq N, m \geq 1$ and $\frac{m}{n} \notin \mathbb{N}$. Let $d = \gcd(m, n)$, $m_1 = \frac{m}{d}$, $n_1 = \frac{n}{d}$ (by hypothesis $n_1 \geq 2$), then let

$l = \left\lceil \frac{N+1-n}{n_1-1} \right\rceil - 1$ (the ceiling function), $\rho = (N+1-n) - l(n_1-1)$. If $l = 0$ then $\tau = (n-1, N+1-n)$ and $\lambda = (m^{N+1-n}, 0^{n-1})$. If $l \geq 1$ then $\tau = (n-1, (n_1-1)^l, \rho)$, $\mu = n_1-1$, $s = d-1$ and $\lambda = \Lambda(n_1-1, d-1, l, \rho, m_1)$, that is, $\lambda = ((m+lm_1)^\rho, (m+(l-1)m_1)^\mu, \dots, m^\mu, 0^{n-1})$. Note that the first part of τ is always $n-1$ (and λ ends in $n-1$ zeros).

The rational Cherednik algebra \mathbf{A} was investigated by Berest, Chmutova, Etingof, Ginzburg, Guay, Opdam and Rouquier in a series of papers [1],[2],[3],[9],[10]. Here we consider the faithful representation of \mathbf{A} as the algebra generated by $\{\mathcal{D}_i, x_i : 1 \leq i \leq N\} \cup S_N$ of operators on polynomials on \mathbb{R}^N (where x_i denotes the multiplication operator). Suppose $\kappa = \kappa_0$ is a singular value and τ, λ are defined as in the previous section and used in Theorem 7, and let $M_\tau = \text{span}_{\mathbb{P}} \{\zeta_{w\lambda}^x : w \in E_\tau\}$ where \mathbb{P} denotes $\mathbb{Q}[x_1, \dots, x_N]$, the polynomials on \mathbb{R}^N . Then M_τ is a module for the Cherednik algebra \mathbf{A} specialized to $\kappa = \kappa_0$; clearly M_τ is closed under multiplication by polynomials and the action of S_N . It is closed under $\{\mathcal{D}_i : 1 \leq i \leq N\}$, indeed suppose p is a polynomial and $g \in \text{span}_{\mathbb{Q}} \{\zeta_{w\lambda}^x : w \in E_\tau\}$ then by the product rule $\mathcal{D}_i(pg) = p\mathcal{D}_i g + g \frac{\partial}{\partial x_i} p + \kappa \sum_{j \neq i} ((i, j)g) \frac{p(x) - (i, j)p(x)}{x_i - x_j} \in M_\tau$ when $\kappa = \kappa_0$ and $\mathcal{D}_i g = 0$.

It is a plausible conjecture that we have found all the singular polynomials for S_N (perhaps to be settled in a later paper). The structure of κ_0 -critical pairs (see Definition 7) may be worth further investigation, with a view to finding a general algorithm for their construction, and maybe a uniqueness result in the case that $h(\lambda, \kappa+1)$ has a zero of multiplicity 1 at $\kappa = \kappa_0$. Such a result would simplify the argument used here.

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