1. a) $$\omega = \int_{H<E} d^3r\, d^3p$$

with $$H(\vec{r},\vec{p}) = \frac{p^2}{2m} - \frac{\gamma}{r}$$

Boundaries on integral:

$$\frac{p^2}{2m} - \frac{\gamma}{r} < E < 0$$

Momentum $p = |\vec{p}|$ takes any value from 0 to ∞

Then for given p, $$\frac{\gamma}{r} > \frac{p^2}{2m} - E$$

$$0 < r < \frac{\gamma}{\frac{p^2}{2m} - E} = \frac{2m\gamma}{p^2 + 2m|E|} \equiv R$$

So $$\omega = \int_0^\infty 4\pi p^2 dp \int_0^R 4\pi r^2 dr$$

$$= \frac{16\pi^2}{3} \int p^2 R^3\, dp$$

$$= \frac{16\pi^2}{3} \int_0^\infty p^2 \frac{8m^3\gamma^3}{(p^2 + 2m|E|)^3}\, dp$$

$$= \frac{16\pi^2}{3} (2m\gamma)^3 \int_0 ^\infty \frac{p^2}{(p^2+a^2)^3}\, dp$$

for $a = \sqrt{2m|E|}$

Look up integral $$\int_0^\infty \frac{p^2}{(p^2+a^2)^3}\, dp = \frac{1}{a^3} \frac{\pi}{16}$$

So $\boxed{\omega = \frac{\pi^3}{3}\left(\frac{2m\gamma^2}{|E|}\right)^{3/2}}$

Corresponds to a number of states

$$\Sigma_{cl} = \frac{\omega}{\omega_0} = \frac{\omega}{h^3} = \frac{\omega}{(2\pi\hbar)^3}$$

$$= \frac{1}{24}\left(\frac{2m\gamma^2}{\hbar^2|E|}\right)^{3/2} = \frac{1}{3}\left(\frac{m\gamma^2}{2\hbar^2|E|}\right)^{3/2}$$

b) For quantum calculation, have

$$\Sigma_{Q} = \sum_{n=1}^{n_0} n^2$$

where $\mathcal{E}_{n_0} = E$

Look up $\sum_{n=1}^{n_0} n^2 = \frac{1}{6}(n_0)(n_0+1)(2n_0+1)$

(or get from my formula)

We have $\frac{m\gamma^2}{2\hbar^2 n_0^2} = |E|$

$$n_0 = \left(\frac{m\gamma^2}{2\hbar^2|E|}\right)^{1/2}$$

For large n_0, $\Sigma_Q \to \frac{1}{3} n_0^3$

$$= \frac{1}{3}\left(\frac{m\gamma^2}{2\hbar^2|E|}\right)^{3/2} = \Sigma_{cl}$$

as expected in classical limit

2. Have $Q_N = \frac{1}{N!} Q_1^N$ (atoms are identical)

$$Q_1 = \frac{1}{h^3}\int d\omega\, e^{-\beta H}$$

$$H = \frac{p^2}{2m} + cV(\vec{r})$$

$$= \frac{1}{h^3}\int_0^\infty 4\pi p^2 e^{-\frac{\beta p^2}{2m}} dp \int d^3r\, e^{-\beta c V(\vec{r})}$$

$$= \frac{4\pi}{h^3}\left(\frac{2m}{\beta}\right)^{3/2} \underbrace{\int_0^\infty u^2 e^{-u^2} du}_{\frac{\sqrt{\pi}}{4}} \int d^3r\, e^{-\beta c V}$$

$$= \underbrace{\left(\frac{2\pi m}{h^2 \beta}\right)^{3/2}}_{\frac{1}{\Lambda^3}} \int d^3r\, e^{-\beta c V}$$

Consider space integral: $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, e^{-\beta c\sqrt{x^2+y^2+4z^2}}$

Define $z' = 2z$

$$\rightarrow \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz'\, e^{-\beta c\sqrt{x^2+y^2+z'^2}}$$

$$= \frac{1}{2}\int_0^\infty 4\pi r'^2 dr'\, e^{-\beta c r'}$$

$$= 2\pi \frac{1}{(\beta c)^3} \underbrace{\int_0^\infty u^2 e^{-u} du}_{=2}$$

$$= \frac{4\pi}{(\beta c)^3}$$

So can write $\boxed{Q_1 = \frac{4\pi}{(\Lambda\beta c)^3}}$

$$\boxed{Q_1 = 4\pi \left(\frac{1}{\beta c}\right)^3 \left(\frac{2\pi m}{h^2\beta}\right)^{3/2}}$$

Then $A = -kT \ln Q_N$

$$= -kT\left[N \ln Q_1 - N \ln N + N\right]$$

$$= -NkT \left\{ \ln\left[\frac{4\pi}{\beta^3 c^3}\left(\frac{2\pi m}{h^2\beta}\right)^{3/2}\frac{1}{N}\right] + 1 \right\}$$

$$U = \frac{\partial}{\partial\beta}(\beta A)$$

$$= -N \frac{\partial}{\partial\beta}\left\{ \ln \beta^{-9/2} + \text{const...} \right\}$$

$$= \frac{9}{2} N \frac{1}{\beta} = \boxed{\frac{9}{2} N k_B T}$$

Energy per particle $\boxed{\frac{U}{N} = \frac{9}{2} k_B T}$

and entropy $S = -\frac{\partial A}{\partial T}$

$$= Nk\left\{ \ln\left[\frac{4\pi}{N}\left(\frac{kT}{c}\right)^3\left(\frac{2\pi m kT}{h^2}\right)^{3/2}\right] + 1 \right\}$$

$$+ NkT\left\{ \frac{9}{2}\frac{1}{T} \right\}$$

$$\boxed{S = Nk\left\{ \ln\left[\frac{4\pi}{N}\left(\frac{kT}{c}\right)^3\left(\frac{2\pi m kT}{h^2}\right)^{3/2}\right] + \frac{11}{2} \right\}}$$

3. a) Here $Q_N = Q_1^N$ (distinguishable)

and $Q_1 = 1 + e^{-\beta\varepsilon}$ ... only two states in sum

So $A = -kT \ln Q_N$

$$= -NkT \ln(1 + e^{-\beta\varepsilon})$$

$$U = \frac{\partial}{\partial\beta}(\beta A) = -N \frac{\partial}{\partial\beta} \ln(1 + e^{-\beta\varepsilon})$$

$$= -N \frac{-\varepsilon e^{-\beta\varepsilon}}{1 + e^{-\beta\varepsilon}}$$

$$\boxed{U = \frac{N\varepsilon}{1 + e^{\beta\varepsilon}}}$$

$$S = -\frac{\partial A}{\partial T} = Nk \ln(1 + e^{-\beta\varepsilon})$$

$$- NkT \frac{\left(-\frac{\varepsilon}{kT^2}\right) e^{-\beta\varepsilon}}{1 + e^{-\beta\varepsilon}}$$

$$\boxed{S = Nk \left\{ \ln(1 + e^{-\beta\varepsilon}) + \frac{\beta\varepsilon}{1 + e^{\beta\varepsilon}} \right\}}$$

$$C = \frac{\partial U}{\partial T} = -\frac{N\varepsilon\left(-\frac{\varepsilon}{kT^2} e^{\beta\varepsilon}\right)}{(1 + e^{\beta\varepsilon})^2}$$

$$\boxed{C = Nk \frac{(\beta\varepsilon)^2 e^{\beta\varepsilon}}{(1 + e^{\beta\varepsilon})^2}}$$

b) For $kT \ll \varepsilon$, $\beta\varepsilon \gg 1$

$$U \to N\varepsilon e^{-\beta\varepsilon}$$

$$S \to Nk\{ e^{-\beta\varepsilon} + \beta\varepsilon e^{-\beta\varepsilon}\} = Nk(1+\beta\varepsilon)e^{-\beta\varepsilon}$$

$$C \to Nk(\beta\varepsilon)^2 e^{-\beta\varepsilon}$$

All three go exponenticlly to zero
This makes sense because for $kT \ll \varepsilon$,
expect excitation to be "frozen out"

All particles in ground state, so no
energy, entropy, or heat capacity

For $kT \gg \varepsilon$, $\beta\varepsilon \ll 1$

$$U \to \frac{N\varepsilon}{2}$$

$$S \to Nk_B\{\ln 2 + \frac{\beta\varepsilon}{2}\} \approx Nk_B \ln 2$$

$$C \to Nk \frac{(\beta\varepsilon)^2}{4} \to 0$$

Here expect $N/2$ particles in each state,
so U makes sense

Entropy is $k_B \ln 2^N$, where $2^N = \Omega$
Ω = # of distinct arrangements for both states
equally likely.

And $C \to 0$ since U can't increase further

4. Total energy E means that there are

$$n = \frac{E}{\varepsilon} \quad \text{particles in excited state}$$
$$N-n \quad \text{in ground state}$$

Particles are distinguishable, so there are

$$\Omega = \frac{N!}{n_0!\,n_1!} = \frac{N!}{n!\,(N-n)!}$$

different states of the system that have this energy.

So $S = k \ln \Omega$

$$= k\left[N\ln N - N - n\ln n + n - (N-n)\ln(N-n) + N-n\right]$$

* Need to assume $N-n \gg 1$, true for large enough N at any finite T

$$= k\left[N\ln N - (N-n)\ln(N-n) - n\ln n\right]$$

$$= k\left[N\ln\frac{N}{N-n} - n\ln\frac{n}{N-n}\right]$$

$$= Nk\left[\ln\frac{1}{1-\frac{n}{N}} - \frac{n}{N}\ln\frac{\frac{n}{N}}{1-\frac{n}{N}}\right]$$

$$= Nk\left[-\ln\left(1-\frac{n}{N}\right) + \frac{n}{N}\ln\left(\frac{N}{n}-1\right)\right]$$

$$\boxed{S = Nk\left[-\ln\left(1-\frac{E}{N\varepsilon}\right) + \frac{E}{N\varepsilon}\ln\left(\frac{N\varepsilon}{E}-1\right)\right]}$$

Then $$\frac{1}{T} = \frac{\partial S}{\partial E} = Nk\left\{\frac{\frac{1}{N\varepsilon}}{1-E/N\varepsilon} + \frac{1}{N\varepsilon}\ln\left(\frac{N\varepsilon}{E}-1\right) + \frac{E}{N\varepsilon}\,\frac{-N\varepsilon/E^2}{\frac{N\varepsilon}{E}-1}\right\}$$

$$\frac{1}{T} = Nk\left\{ \frac{1}{N\varepsilon - E} + \frac{1}{N\varepsilon}\ln\left(\frac{N\varepsilon}{E} - 1\right) - \frac{1}{N\varepsilon - E} \right\}$$

$$\boxed{\frac{1}{T} = \frac{k}{\varepsilon}\ln\left(\frac{N\varepsilon}{E} - 1\right)}$$

So $\frac{N\varepsilon}{E} - 1 = e^{\beta\varepsilon}$ $\qquad \beta = \frac{1}{kT}$

$$\frac{E}{N\varepsilon} = \frac{1}{e^{\beta\varepsilon} + 1}$$

$$\boxed{E = \frac{N\varepsilon}{1 + e^{\beta\varepsilon}}}$$ Same as in 3.

Then also

$$S = Nk\left\{ -\ln\left(1 - \frac{1}{1+e^{\beta\varepsilon}}\right) + \left(\frac{1}{1+e^{\beta\varepsilon}}\right)\beta\varepsilon \right\}$$

$$= Nk\left\{ -\ln\left(\frac{e^{\beta\varepsilon}}{1+e^{\beta\varepsilon}}\right) + \frac{\beta\varepsilon}{1+e^{\beta\varepsilon}} \right\}$$

$$\boxed{S = Nk\left\{ \ln\left(1 + e^{-\beta\varepsilon}\right) + \frac{\beta\varepsilon}{1+e^{\beta\varepsilon}} \right\}}$$ Same as in 3