Lecture 26

Wrap up discussion of Fermi gases

Say a bit about white dwarf stars

= ball of hot, gravitationally bound He atoms

= possible endpoint of stellar fusion

Typical temperature $\sim 10^7$ K

$\sim 10^3$ eV

so atoms all ionized,

Treat as electron gas!

Typical density $\sim 10^{36}\ e^-/m^3$

$$\Rightarrow \quad \varepsilon_F = \left(\frac{3n}{8\pi}\right)^{2/3} \times \frac{h^2}{2m}$$

$$\sim 10^5 \text{ eV}$$

Since $T \ll T_F$, gas is degenerate.

Further, see $\varepsilon_F \sim mc^2 = 511$ keV

So gas is relativistic

Work out this case:

$$N = 2 \cdot \frac{V}{h^3} \int_0^{p_F} 4\pi p^2 \, dp \qquad p_F = \text{Fermi momentum}$$

$$= \frac{8\pi V}{3h^3} p_F^3 \qquad \text{(same for relativistic or not)}$$

$$\Rightarrow \quad p_F = \left(\frac{3n}{8\pi}\right)^{1/3} h$$

Have energy $\varepsilon = mc^2 \left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2}$

So Fermi energy is $\varepsilon_F = mc^2(1+x_F^2)^{1/2}$

$$x_F \equiv \frac{p_F}{mc}$$

But now U is more complicated!

$$U = \frac{8\pi V}{h^3}\int_0^{p_F} mc^2\left(1+\frac{p^2}{m^2c^2}\right)^{1/2} p^2\,dp$$

$$= 8\pi V\frac{m^4c^5}{h^3}\int_0^{x_F}(1+x^2)^{1/2}x^2\,dx$$

Can look that integral up:

$$U = \pi V\frac{m^4c^5}{h^3}\left[x_F(1+x_F^2)^{1/2}(1+2x_F^2) - \sinh^{-1}x_F\right]$$

From this we can get pressure:

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$$

note S is fixed as long as we stay at $T=0$

But V appears twice: in $x_F = \left(\frac{3N}{8\pi V}\right)^{1/3}\frac{h}{mc}$ too

Easiest to use integral form:

$$P = -8\pi\frac{m^4c^5}{h^3}\left[\int_0^{x_F}(1+x^2)^{1/2}x^2\,dx + V\frac{\partial x_F}{\partial V}x_F^2(1+x_F^2)^{1/2}\right]$$

$$\frac{\partial x_F}{\partial V} = -\frac{1}{3}\frac{x_F}{V}$$

So

$$P = -\pi\frac{m^4c^5}{h^3}\left[x_F(1+x_F^2)^{1/2}(1+2x_F^2) - \sinh^{-1}x_F - \frac{8}{3}x_F^3(1+x_F^2)^{1/2}\right]$$

$$P = \pi \frac{m^4c^5}{h^3} \left\{ x_F(1+x_F^2)^{1/2}\left[\frac{8}{3}x_F^2 - 1 - 2x_F^2\right] + \sinh^{-1}x_F \right\}$$

$$\left[\frac{2}{3}x_F^2 - 1\right]$$

$$P = \underbrace{\frac{\pi}{3}\frac{m^4c^5}{h^3}}_{\sim 6\times 10^{21}\ \mathrm{Pa}} \Big[\underbrace{x_F(1+x_F^2)^{1/2}(2x_F^2-3) + 3\sinh^{-1}x_F}_{\equiv A(x_F)} \Big]$$

Large positive pressure, as might expect

Should be balanced by gravity

- really, gas would be inhomogeous ... complicated

Can get rough model' by setting

$$P \approx \frac{dE_{grav}}{dV} = \frac{1}{4\pi R^2}\frac{d}{dR}\left[-\frac{3}{5}\frac{GM^2}{R}\right]$$

$E_{grav} = -\frac{3}{5}\frac{GM^2}{R}$

$$= \frac{GM^2}{4\pi R^4}$$

for M = total mass of star

So

$$\frac{\pi}{3}\frac{m^4c^5}{h^3} A(x) = \frac{3}{5}\frac{GM^2}{4\pi R^4}$$

$$\text{for } x = \left(\frac{3N}{8\pi V}\right)^{1/3}\frac{h}{mc}$$

Since star made of ^4He, have equal # electrons, protons, & neutrons

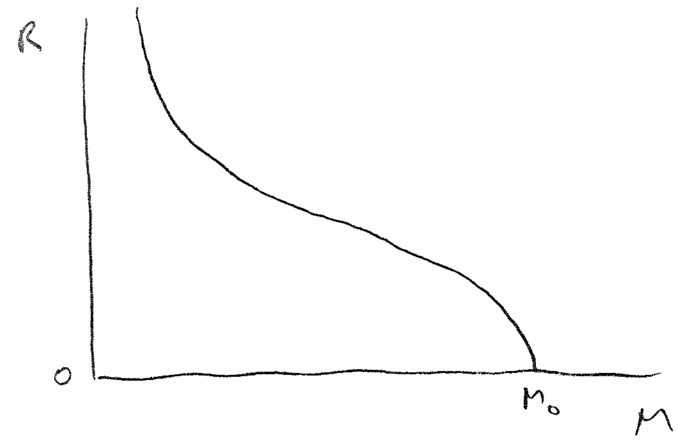
$$\text{So } N = \frac{M}{m_p + m_n + m_e} \approx \frac{M}{2m_p}$$

$$\text{also } V = \frac{4}{3}\pi R^3$$

So $\quad x = \left(\frac{9M}{64\pi^2 m_p}\right)^{1/3} \frac{h}{mcR} = \left(\frac{9\pi M}{m_p}\right)^{1/3} \frac{\hbar}{mcR}$

So our pressure-gravity equation ultimately relates M and R

$\Rightarrow$ implicitly gives $R(M)$



Find $R \to 0$ at finite $M = M_0$

for $M > M_0$, Fermi pressure is unable to prevent collapse

In our simple model, use

$$A \to 2x^2(x^2-1) \quad \text{for } x \gg 1 \quad (R \to 0)$$

Gives
$$R \simeq \frac{(9\pi)^{1/3}}{2} \frac{\hbar}{mc} \left(\frac{M}{m_p}\right)^{1/3} \left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2}$$

for
$$M_0 = \frac{15}{64}\sqrt{5\pi}\, \frac{(\hbar c/G)^{3/2}}{m_p^2}$$

$\approx 10^{30}$ kg

$\approx$ mass of our sun

(Remember, this is all rough since we assumed a homogeneous gas!)

For $M > M_0$, can't have white dwarf

Instead, electrons + protons → neutrons,

get neutron star

But this is again sustained by Fermi pressure

Also unstable for large enough M: M_0'

and $M_0' \sim M_0$, since M_0 doesn't

depend on particle mass

For larger mass yet, collapse to black hole!

Don't really know how to do stat mech then,

since we don't have a microscopic model.

Want to wrap everything up with a somewhat

different way of approaching stat mech

We've focused on ensemble theory:

get system properties from averaging

But can also start from kinetic theory:

try to model how microstate evolves in time

look for stationary solutions

= equilibrium states

Generally hard, but allows study of

non-equilibrium systems too

Just give an idea of how it works:

Imagine <u>nearly</u> ideal gas

Particles occupy single particle states ψ_i

But collisions permit states to change

For instance, could have particles in states 1 + 2 collide and produce states 3 & 4

Process occurs at rate

$$R_{12\to 34} = \varkappa n_1 n_2$$

$\varkappa$ = rate constant

n_i = # particles in state i

of course, need $\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4$, conserve energy

In equilibrium, state populations don't change

But collisions still occur...

Must have $R_{12\to 34} = R_{34\to 12}$

"principle of detailed balance"

Have $R_{34\to 12} = \varkappa n_3 n_4$

Same $\varkappa$: reversibility of microscopic laws

Means $n_1 n_2 = n_3 n_4$

whenever $\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4$

or $n(\varepsilon_1) n(\varepsilon_2) = n(\varepsilon_1 + a)\, n(\varepsilon_2 - a)$

Define $y(\varepsilon) = \ln n(\varepsilon)$:

$$y(\varepsilon_1) + y(\varepsilon_2) = y(\varepsilon_1 + a) + y(\varepsilon_2 - a)$$

$$\lim_{a \to 0} \quad \frac{y(\varepsilon_2) - y(\varepsilon_2 - a)}{a} = \frac{y(\varepsilon_1 + a) - y(\varepsilon_1)}{a}$$

$$y'(\varepsilon_2) = y'(\varepsilon_1) \qquad \text{for any } \varepsilon_1, \varepsilon_2$$

$$\Rightarrow \quad \frac{dy}{d\varepsilon} = \text{constant}$$

$$\text{so} \quad y = \alpha - \beta\varepsilon = \ln n(\varepsilon)$$

$z = e^{\alpha}$

$$\boxed{n(\varepsilon) = z e^{-\beta\varepsilon}} \qquad \text{Boltzmann distribution!}$$

Use this to do averages, get all the rest ...

But what about quantum statistics?

Need to modify collision rate

For bosons, QM says we get Bose enhancement:

$$R_{12 \to 34} = \alpha\, n_1 n_2 (1 + n_3)(1 + n_4)$$

$$R_{34 \to 12} = \alpha\, n_3 n_4 (1 + n_1)(1 + n_2)$$

So now have

$$n_1 n_2 (1 + n_3)(1 + n_4) = n_3 n_4 (1 + n_1)(1 + n_2)$$

$$\frac{n_1}{1+n_1} \frac{n_2}{1+n_2} = \frac{n_3}{1+n_3} \frac{n_4}{1+n_4}$$

So define $y = \ln \frac{n}{1+n}$:

Get $y(\varepsilon_1) + y(\varepsilon_2) = y(\varepsilon_3) + y(\varepsilon_4)$, as before

so $y(\varepsilon) = \alpha - \beta\varepsilon$

$$\frac{n}{1+n} = e^{\alpha - \beta\varepsilon}$$

$$1 + \frac{1}{n} = e^{-\alpha + \beta\varepsilon}$$

$$\frac{1}{n} = e^{-\alpha + \beta\varepsilon} - 1$$

$$\boxed{n(\varepsilon) = \frac{1}{z^{-1} e^{\beta\varepsilon} - 1}}$$

Bose-Einstein distribution

Fermions are similar:

get $R_{12\to34} \propto n_1 n_2 (1-n_3)(1-n_4)$

no collisions if ψ_3, ψ_4 are occupied

Similar calculation gives

$$\boxed{n(\varepsilon) = \frac{1}{z^{-1} e^{\beta\varepsilon} + 1}}$$

Fermi-Dirac

Final

- Memorize thermo relations
- Know basic definitions of stat mech
  (Partition functions + how to get thermo quantities from them)
- I'll give any non-elementary integrals
- Probably have one "derivation" on exam

A few problems for review:

1.11, 1.15, 1.16
3.10, 3.38, 3.40
4.7, 6.8
7.19, 7.27, 8.1, 8.14

Course overall:

Opinions?

My thoughts:

- liked doing thermo first, but a book would help
- spend less time on ensemble theory, more on applications
- want to get to interacting systems (mean field theory)