Lecture 25

Today talk about magnetism in a Fermi system.

Looks at magnetism in classical systems before: Lecture 16
§ 3.9

Langevin paramagnetism:

Classical spins

$$M = n\mu^* L(\beta\mu^*\mathcal{H})$$

Use μ^* to distinguish from μ = chem potential

n = particle density N/V

μ^* = magnetic moment of particle

M = magnetization (net moment/volume)

H = applied field

$\beta = 1/kT$

$$L(x) = \coth x - \frac{1}{x}$$

Gives susceptibility $\chi = \left.\frac{\partial M}{\partial \mathcal{H}}\right|_{H=0} = \frac{n\mu^{*2}}{3kT}$

Curie law

Quantized spins

$$M = n\mu^* B_J(\beta\mu^*\mathcal{H})$$

J = angular momentum quantum number

$\mu^* = g_J\mu_B J$ $\quad g_J$ = g-factor

$$B_J(x) = \left(1+\frac{1}{2J}\right)\coth\left[\left(1+\frac{1}{2J}\right)x\right] - \frac{1}{2J}\coth\frac{x}{2J}$$

Gives $\chi = \frac{n\mu^{*2}}{3kT}\,\frac{J+1}{J}$

Note $\chi \to \infty$ as $T \to 0$

All spins line up with field

In a degenerate Fermi system, more complicated

Low energy spin states all filled...

only particles near Fermi energy can flip

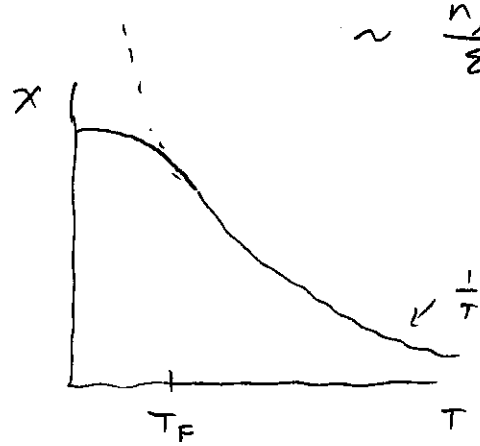
We know # of particles with states available

is $O\left(\frac{kT}{\varepsilon_F}\right)$

So expect that $\chi \to \chi_{class} \times \frac{kT}{\varepsilon_F}$

$\sim \frac{n\mu^{*2}}{\varepsilon_F}$ constant, indep T

Guess:



First worked out by Pauli: Pauli paramagnetism

Book derivation follows Pauli's, but is unnecessarily complicated

I like this one better:

Consider gas of electrons

$$H = \frac{p^2}{2m} - \vec{\mu} \cdot \vec{\mathcal{H}}$$

eigen energies $\varepsilon_{p\sigma} = \frac{p^2}{2m} - \sigma \mu^* \mathcal{H}$

$\sigma = \pm 1$

$\mu^* = g\mu_B J = 2 \cdot \mu_B \times \frac{1}{2} = \mu_B$

for electrons

Know $\quad n_{p\sigma} = \dfrac{1}{z^{-1} e^{\beta \varepsilon_{p\sigma}} + 1}$

So number of spin-up particles is

$$N_+ = \sum_{\vec{p}} n_{p+}$$

$$\rightarrow \frac{V}{h^3} \int n_{p+} \, d^3p$$

$$\rightarrow \frac{2\pi V}{h^3} (2m)^{3/2} \int \frac{1}{z^{-1} e^{\beta(\varepsilon - \mu^* \mathcal{H})} + 1} \sqrt{\varepsilon} \, d\varepsilon$$

Define $\quad z_+ = z e^{\beta \mu^* \mathcal{H}}$

Then $$N_+ = \frac{2\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{\sqrt{\varepsilon}}{z_+^{-1} e^{\beta\varepsilon} + 1} \, d\varepsilon$$

$$= \frac{V}{\Lambda^3} f_{3/2}(z_+)$$

Similarly, # spin down is

$$N_- = \frac{V}{\Lambda^3} f_{3/2}(z_-)$$

for $\quad z_- = z e^{-\beta \mu^* \mathcal{H}}$

The magnetization is then

$$M = \frac{1}{V} \mu^* (N_+ - N_-) = \text{net moment/volume}$$

$$\boxed{M = \frac{\mu^*}{\Lambda^3} \left[f_{3/2}(z_+) - f_{3/2}(z_-) \right]}$$

So given routine to calculate $f_{3/2}$, get $M(N, H, T)$

Look at susceptability $\chi = \left.\frac{\partial M}{\partial \mathcal{H}}\right|_{\mathcal{H}=0}$

$$\chi = \frac{\mu^*}{\Lambda^3}\left[\frac{\partial f_{3/2}(z_+)}{\partial z_+} \frac{\partial z_+}{\partial \mathcal{H}} - \frac{\partial f_{3/2}(z_-)}{\partial z_-} \frac{\partial z_-}{\partial \mathcal{H}} \right]\Bigg|_{\mathcal{H}=0}$$

Have derivitive relation $\frac{\partial f_{3/2}}{\partial z} = \frac{1}{z} f_{1/2}$

and $\frac{\partial z_+}{\partial \mathcal{H}} = \frac{\partial}{\partial \mathcal{H}} z e^{\beta \mu^* \mathcal{H}}$

$= \beta \mu^* z_+$

$\frac{\partial z_-}{\partial \mathcal{H}} = -\beta \mu^* z_-$

So $\chi = \frac{\mu^*}{\Lambda^3} \beta \mu^* \left[f_{1/2}(z_+) + f_{1/2}(z_-) \right]\Big|_{\mathcal{H}=0}$

$$\boxed{\chi = \frac{2\mu^{*2}}{\Lambda^3} \beta f_{1/2}(z)}$$ – General result

Look at limits

$T \to \infty$ $\quad f_{1/2}(z) \to z$

or $z \to 0$

also $N = \frac{gV}{\Lambda^3} f_{3/2}(z) \to \frac{gV}{\Lambda^3} z$

so $z \to \frac{n\Lambda^3}{g}$ $\quad$ Here $g = 2s+1$
$= 2$

So $\chi \to \frac{2\mu^{*2}}{\Lambda^3} \beta \times \frac{n\Lambda^3}{g}$

$$\chi \to \frac{n\mu^{*2}}{kT}$$

Compare Brillouin formula for $J=\frac{1}{2}$:

$$\chi_{class} = \frac{n\mu^{*2}}{3kT} \times \frac{\frac{1}{2}+1}{\frac{1}{2}}$$

$$= \frac{n\mu^{*2}}{kT} \quad \text{agrees}$$

Degenerate limit $T \to 0$

$z \to \infty$

Use $$f_\nu(z) \to \frac{(\ln z)^\nu}{\Gamma(\nu+1)} = \frac{(\beta\mu)^\nu}{\Gamma(\nu+1)}$$

$\mu \to \varepsilon_F$

$$\chi \to 2\mu^{*2}\left(\frac{2\pi m kT}{h^2}\right)^{3/2}\left(\frac{1}{kT}\right)\left(\frac{\varepsilon_F}{kT}\right)^{1/2}\frac{1}{\frac{1}{2}\sqrt{\pi}}$$

$$= \mu^{*2}\, 4\pi\left(\frac{2m}{h^2}\right)^{3/2}\varepsilon_F^{1/2}$$

Now use $$\varepsilon_F = \left(\frac{3N}{8\pi V}\right)^{2/3}\frac{h^2}{2m}$$

$$\Rightarrow \quad \frac{2m}{h^2} = \left(\frac{3n}{8\pi}\right)^{2/3}\frac{1}{\varepsilon_F}$$

and $$\chi \to \mu^{*2}\, 4\pi\left(\frac{3n}{8\pi}\right)\frac{\varepsilon_F^{1/2}}{\varepsilon_F^{3/2}} = \boxed{\frac{3}{2}\frac{\mu^{*2} n}{\varepsilon_F}}$$

Agrees with expectation $\chi \sim \frac{n\mu^{*2}}{\varepsilon_F}$ ✓

Compare to experiment: χ for metals

Do see $\chi \to$ const as $T \to 0$

But constant not quite right

Really need to account for another effect:

Landau diamagnetism

Aside from spin, can also get magnetism from motion of electrons

Charged particles, so circular motion $\Rightarrow$ magnetic moment

Turns out, this effect is comparable to paramagnetic one

Treat using Landau levels

Energy in field $\mathcal{H}\hat{z}$ is

$$\varepsilon_{pj} = \frac{p_z^2}{2m} + \underbrace{\frac{e\hbar\mathcal{H}}{mc}\left(j+\frac{1}{2}\right)}_{2\mu_B\mathcal{H}\left(j+\frac{1}{2}\right)} \qquad j=0,1,2,\ldots$$

$$\mu_B = \frac{e\hbar}{2mc}$$

Derive this in quantum, take as given here.

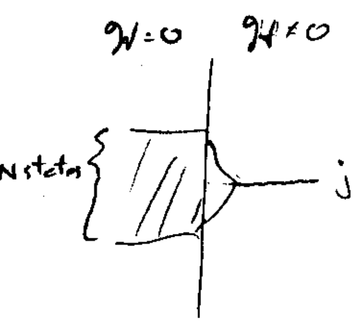
Each level $(p_z\, j)$ is highly degenerate

Transverse energy $\frac{p_x^2+p_y^2}{2m} \rightarrow 2\mu_B\mathcal{H}\left(j+\frac{1}{2}\right)$

# of states doesn't change, so degeneracy of Landau level j

= # of states with

$$2\mu_B\mathcal{H}j \le \frac{p_x^2+p_y^2}{2m} \le 2\mu_B\mathcal{H}(j+1)$$



Can count states using $\frac{1}{h^2}\int dx\,dy\,dp_x\,dp_y$

$$= \frac{L_x L_y}{h^2}\int_{\sqrt{4m\mu_B\mathcal{H}j}}^{\sqrt{4m\mu_B\mathcal{H}(j+1)}} 2\pi p\,dp$$

L_x, L_y = size of system

$$= \frac{\pi L_x L_y}{h^2}\left[4m\mu_B\mathcal{H}(j+1) - 4m\mu_B\mathcal{H}j\right]$$

$$= \frac{4\pi m\mu_B\mathcal{H}}{h^2} L_x L_y$$

$$= \frac{e\mathcal{H}}{hc} L_x L_y \qquad \text{using } \mu_B$$

Then we can get

$$\ln\mathcal{Q} = \sum_{p_z j} \ln\left(1 + z e^{-\beta\varepsilon_{pj}}\right) \times 2$$

spin degeneracy

$\sum_p \to \int \frac{L_z}{h}\,dp$, so

$$\ln\mathcal{Q} \to 2\int_{-\infty}^{\infty} \frac{L_z}{h}\,dp \sum_{j=0}^{\infty}\left(L_x L_y \frac{e\mathcal{H}}{hc}\right)\ln\left\{1 + z e^{-\beta\left[\frac{p^2}{2m} + 2\mu_B\mathcal{H}(j+\frac{1}{2})\right]}\right\}$$

$$= \frac{2Ve\mathcal{H}}{h^2 c}\int_{-\infty}^{\infty} dp \sum_j \ln\left\{1 + z e^{-\beta[\quad]}\right\}$$

Need to be a little careful with evaluation,

See that as $\mathcal{H}\to 0$, $\ln\mathcal{Q} \to 0\times\infty$

And we will take $\mathcal{H}\to 0$ to get χ

Get right answer use Euler formula for sum

$$\sum_{j=0}^{\infty} f(j+\tfrac{1}{2}) \approx \int_0^\infty f(x)\,dx + \frac{1}{24} f'(0)$$

So $$\ln \mathcal{Q} \to \frac{gVe\mathcal{H}}{h^2 c}\left\{ \int_0^\infty dx \int_{-\infty}^{\infty} dp \, \ln\left[1 + z e^{-\beta\left(\frac{p^2}{2m} + 2\mu_B \mathcal{H} x\right)}\right] \right.$$

$$\left. - \frac{1}{12}\beta \mu_B \mathcal{H} \int_{-\infty}^{\infty} \frac{dp}{z^{-1} e^{\beta(p^2/2m)} - 1} \right\}$$

First term is independent of $\mathcal{H}$

see by defing $x' = \mathcal{H} x$

$\mathcal{H}$ goes away

Second term gives

$$\ln \mathcal{Q} = -\frac{2\pi V (2m)^{3/2}}{3h^3} (\mu_B \mathcal{H})^2 \beta^{1/2} \underbrace{\int_0^\infty \frac{y^{-1/2}\,dy}{z^{-1}e^{y} + 1}}_{\sqrt{\pi}\, f_{1/2}(z)}$$

From this, can get

$$\langle M \rangle = \frac{1}{V} \operatorname{Tr} \hat{\mu}_z \rho = \frac{1}{V} \frac{\sum \mu_z e^{-\beta(H_0 - \mu_z \mathcal{H})}}{\mathcal{Q}}$$

$$= \frac{1}{\beta V} \frac{\partial}{\partial \mathcal{H}} \ln \mathcal{Q}$$

And $$\chi = \frac{\partial \langle M \rangle}{\partial \mathcal{H}} = \frac{1}{\beta V} \frac{\partial^2}{\partial \mathcal{H}^2} \ln \mathcal{Q}$$

$$\chi = \frac{1}{\beta V}\left[- \frac{2V(2\pi m)^{3/2}}{3h^3} \mu_B^2 \beta^{1/2} f_{1/2}(z)\right]$$

$$= -\frac{2}{3}\left(\frac{2\pi m kT}{h^2}\right)^{3/2} \frac{\mu_B^2}{kT} f_{1/2}(z)$$

$$\boxed{\chi_{dia} = -\frac{2}{3\Lambda^3} \frac{\mu_B^2}{kT} f_{1/2}(z)}$$

See $\chi < 0$: diamagnetic

substance repelled from field

Otherwise, looks like Pauli paramagnetism

$$\chi_{para} = \frac{2}{\Lambda^3} \frac{\mu^{*2}}{kT} f_{1/2}(z)$$

μ^* = moment associated with spin

= μ_B for elections

So $\chi_{dia} = -\frac{1}{3}\chi_{para}$

$$\boxed{\chi_{tot} = \frac{4}{3} \frac{\mu^{*2}}{\Lambda^3 kT} f_{1/2}(z)}$$

Reduced a bit from para effect

but same scaling with T etc.

Final

Thursday, April 30

9-12 am, here

Comprehensive, weighted a bit toward later material

Closed book, like midterm

Office hours on Monday & Wednesday next week