Lecture 25

Ideal Fermi gas

Formally, much like Bose gas:

$$\frac{PV}{kT} = \ln \mathcal{Q} = \sum_{\varepsilon} \ln(1 + z e^{-\beta\varepsilon})$$

$$N = \sum_{\varepsilon} \langle n_\varepsilon \rangle = \sum_{\varepsilon} \frac{1}{z^{-1} e^{\beta\varepsilon} + 1}$$

$$\beta = \frac{1}{kT}$$

$$z = e^{\beta\mu}$$

Here any μ allowed

$$0 \le z \le \infty$$

Never more than one particle per state:

No question of condensation

Turn sums into integrals (for non-relativistic gas)

$$\frac{P}{kT} = \frac{g}{\Lambda^3} f_{5/2}(z)$$

$$\frac{N}{V} = \frac{g}{\Lambda^3} f_{3/2}(z)$$

Here g = degeneracy factor

typically $2s+1$ for spin s

$$\Lambda = \sqrt{\frac{h^2}{2\pi m kT}} \quad \text{as usual}$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}\,dx}{z^{-1}e^x + 1} = z - \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} - \dots$$

"Fermi-Dirac functions"

Develop thermodynamics much like Bose gas:

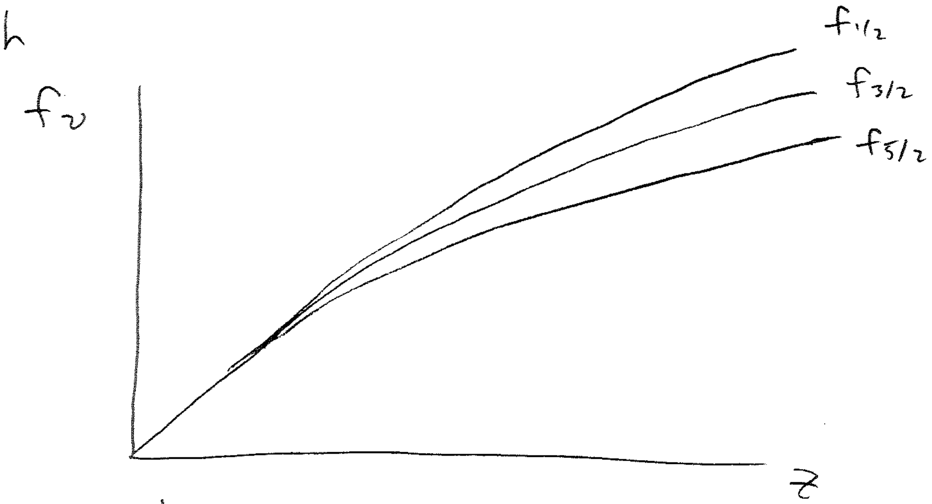
$$U = \frac{3}{2} NkT \frac{f_{5/2}(z)}{f_{3/2}(z)} = \frac{3}{2} PV$$

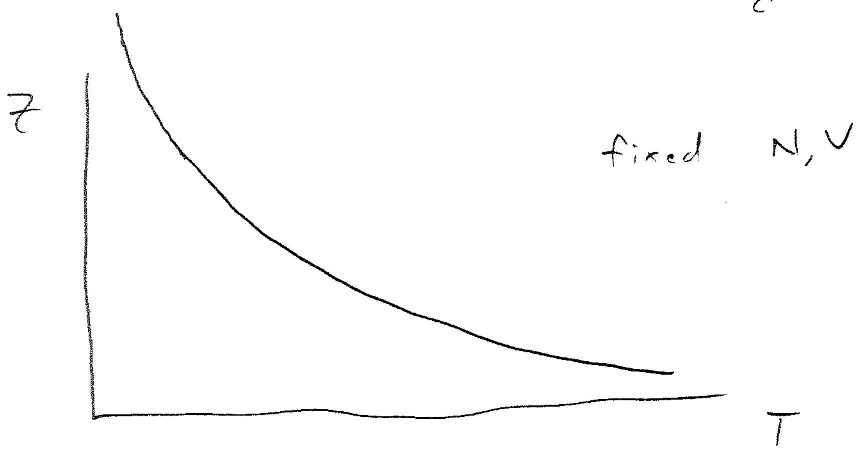
$$\frac{C_V}{Nk} = \frac{15}{4} \frac{f_{5/2}(z)}{f_{3/2}(z)} - \frac{9}{4} \frac{f_{3/2}(z)}{f_{1/2}(z)}$$

$$S = Nk \left\{ \frac{5}{2} \frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z \right\}$$

Sketch

$$\boxed{f_\nu \to \frac{(\ln z)^\nu}{\Gamma(\nu+1)}}$$





Note $P > P_{classical}$ always

Limits: Small $z \Leftrightarrow \frac{N\Lambda^3}{V} \ll 1$:

$f_\nu \to z$

all properties $\to$ classical gas.

Large z: $\frac{N\Lambda^3}{V} \gg 1$ degenerate Fermi gas

Corresponds to $T \to 0$

Then $n_\varepsilon \to \begin{cases} 1 & \text{for } \varepsilon < \mu \\ 0 & \text{for } \varepsilon > \mu \end{cases}$

Can determine μ from $N = \int a(\varepsilon)\, n_\varepsilon \, d\varepsilon$

Or use $\frac{N\Lambda^3}{Vg} = f_{3/2}(z) \to \frac{(\ln z)^{3/2}}{\Gamma(5/2)} = \frac{(\beta\mu)^{3/2}}{\left(\frac{3\sqrt{\pi}}{4}\right)}$

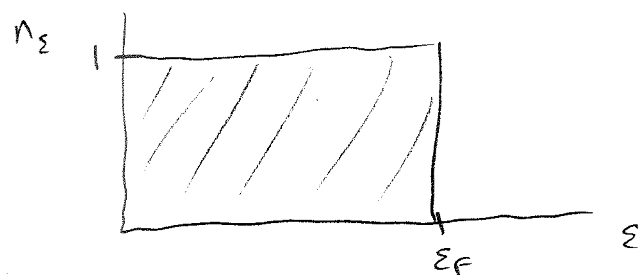
$$\frac{N}{V}\frac{1}{g}\left(\frac{h^2}{2\pi m kT}\right)^{3/2} = \frac{4}{3\sqrt{\pi}}\left(\frac{\mu}{kT}\right)^{3/2}$$

$$\mu = \left(\frac{3}{4\pi}\frac{N}{gV}\right)^{2/3} \cdot \frac{h^2}{2m}$$

$$\boxed{\mu = \left(\frac{6\pi^2 n}{g}\right)^{2/3} \cdot \frac{\hbar^2}{2m} \equiv \varepsilon_F}$$

Fermi energy

= energy of highest occupied state



Total energy of system $= \frac{3}{2} NkT \frac{f_{5/2}(z)}{f_{3/2}(z)}$

$$U = \frac{3}{2} NkT \frac{(\beta\mu)^{5/2}}{\Gamma(\frac{7}{2})} \times \frac{\Gamma(\frac{5}{2})}{(\beta\mu)^{3/2}}$$

$$\Gamma(\tfrac{7}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2})$$

$$= \frac{3}{2} NkT \times \beta\mu \times \frac{2}{5}$$

$$\boxed{E_0 = \frac{3}{5} N \varepsilon_F}$$

Pressure $P_0 = \frac{2}{3} \frac{U_0}{V} = \frac{2}{5} n \varepsilon_F \neq 0$

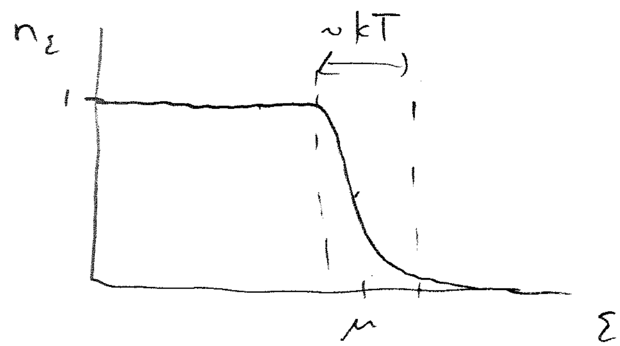
Fermi systems can have substantial energy + pressure as $T \to 0$

But $S = Nk \left\{ \frac{5}{2} \frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z \right\}$

$$= Nk \left\{ \frac{5}{2} \cdot \frac{2}{5} \ln z - \ln z \right\} \to 0$$

as expected

For T small but non-zero, have



Fraction of atoms in excited states $\sim \frac{kT}{\varepsilon_F}$

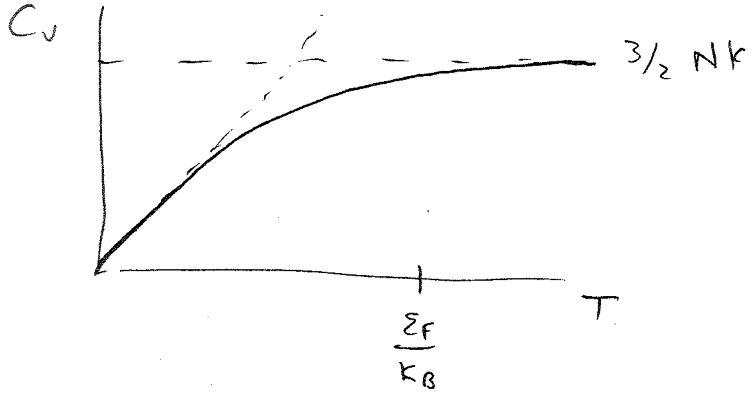
Thermal energies themselves $\sim kT$, so total energy $U \simeq E_0 + O\left(N \frac{(kT)^2}{\varepsilon_F}\right)$

Expect $C_V \propto T$

Can study more quantitatively by using more accurate expansion

$$f_\nu(z) \to \frac{(\ln z)^\nu}{\Gamma(\nu+1)}\left[1 + \nu(\nu-1)\frac{\pi^2}{6}\frac{1}{(\ln z)^2} + \dots\right]$$

Or we can just plot exact C_V:



Get $C_V \approx \frac{\pi^2}{2}\frac{kT}{\varepsilon_F}$ for small T

All this is pretty relevant, because it turns out metals are good example of a degenerate Fermi gas

Metal: crystal lattice of ions with one or more free electrons per ion,

To first approximation, treat electrons as ideal gas

Calculate ε_F for sodium $g = 2$ since $s = \frac{1}{2}$

$$\varepsilon_F = \left(\frac{3N}{8\pi V}\right)^{2/3}\frac{h^2}{2m}$$

$$\frac{N}{V} = 1\ e^- \text{ per ion} \times 2.5\times10^{28}\ m^{-3}$$

Gives $\varepsilon_F = 5 \times 10^{-19}\ J = 3.14\ eV$

Then Fermi temperature $T_F \equiv \frac{\varepsilon_F}{k_B} = 3.64 \times 10^4\ K$

This sets scale for whether temperature is "low"

So room temp corresponds to $T \approx 0$ here

Note $\frac{N\Lambda^3}{V} = \frac{8}{3\sqrt{\pi}} \left(\frac{T_F}{T}\right)^{3/2}$

so $T \lesssim T_F$ corresponds to $\frac{N\Lambda^3}{V} \gtrsim 1$

So we expect specific heat of metals $\propto T$

But that is just electronic contribution

Saw last time (Debye model) that lattice phonons contribute $C_V \propto T^3$

Generally see total

$$C_V = \gamma T + \delta T^3$$

in metals as $T \to 0$

(Note room temp is not cold for lattice... typically $\Theta_D \sim$ room temp)

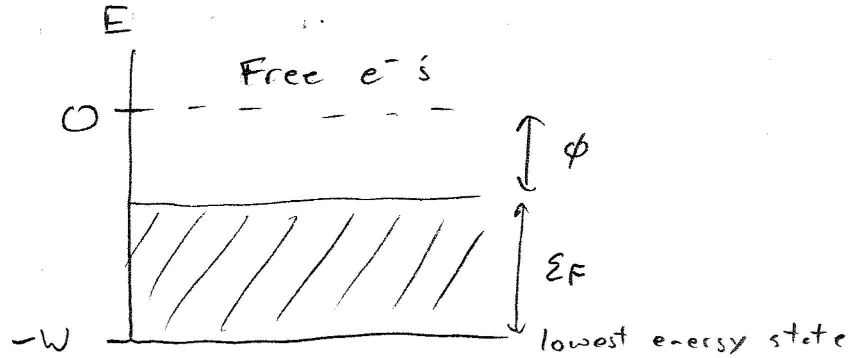
Let's apply model to a process: thermionic emission

Idea: heat metal up until electrons start to fly off. How does emission current depend on T?

Normally, electrons confined by electrostatic attraction of ions

Model as uniform binding potential $U = -W$

Picture



Electrons need thermal energy $\phi = W - \varepsilon_F$ to escape

ϕ = "work function"

How much work it takes to get an electron out

What is emission rate?

In general $R = \dfrac{\text{rate}}{\text{surface area}}$

$$= \int_{p_z > (2mW)^{1/2}} d^3p \; n(\vec{p}) \, u_z$$

where z = normal to surface

$n(\vec{p})\, d^3p = \dfrac{\#\text{ of electrons w/ momentum } \vec{p}}{\text{unit volume}}$

u_z = velocity component $= \dfrac{p_z}{m}$

To get $n(\vec{p})$, note $n(\vec{k})\, d^3k = \dfrac{1}{V} \times \left(\dfrac{L}{2\pi}\right)^3 \times f(\varepsilon_k)\, d^3\vec{k}$

$$= \frac{1}{(2\pi)^3} \frac{1}{e^{(\varepsilon-\mu)/kT}+1} \, d^3k$$

Since $\vec{k} = \frac{\vec{p}}{\hbar}$,

$$n(\vec{p})\, d^3\vec{p} = n(\vec{k})\, d^3\vec{k} \quad \text{(same \# of electrons)}$$

$$= \frac{1}{(2\pi)^3} f(\varepsilon_{\vec{p}}) \frac{d^3\vec{p}}{\hbar^3}$$

So

$$R = 2 \int_{(2mW)^{1/2}}^{\infty} dp_z \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \frac{1}{h^3} \frac{p_z}{m} \frac{1}{e^{(p^2/2m-\mu)/kT}+1}$$

two spin states

Do xy integrals in polar coords, leaves

$$R = \frac{4\pi kT}{h^3} \int_{(2mW)^{1/2}}^{\infty} p_z\, dp_z \ln\left[1 + e^{\left(\mu - \frac{p_z^2}{2m}\right)/kT}\right]$$

$$= \frac{4\pi m kT}{h^3} \int_W^{\infty} d\varepsilon_z \ln\left(1 + e^{(\mu-\varepsilon_z)/kT}\right)$$

Now, unless $T \sim T_F$, have $e^{(\mu - W)/kT} \ll 1$

(if $T \sim T_F$, usually metal is vaporized)

So expand $\ln(1+x) \simeq x$

$$R \to \frac{4\pi m kT}{h^3} \int_W^{\infty} d\varepsilon_z\, e^{(\mu-\varepsilon_z)/kT}$$

$$\boxed{R = \frac{4\pi m k^2 T^2}{h^3} e^{(\mu - W)/kT}}$$

Current density (A/m^2) $J = -eR$

See that we need $kT \sim W - \mu$ to get significant emission, as expected.

But don't actually get to that point: emission always small compared to # of electrons in metal

But do see this temperature dependence
T^2 factor not important
R dominated by $e^{(\mathcal{E}_F - W)/kT}$

Agrees with independent measurements of $\mathcal{E}_F$ & W

Note if they weren't fermions, we'd expect

$$R \sim e^{-W/kT}$$

much smaller!