Lecture 23

Talking about blackbody radiation

Two ways to consider

1) Collection of distinct quantum oscillators (field modes)

Get $\langle n_s \rangle = \dfrac{1}{e^{\beta \hbar \omega_s} - 1}$

2) Collection of indistinguishable photons

Different from ideal Bose gas:

number not conserved

photons easily created, destroyed

No constraint $\sum_s n_s = N$

$\Rightarrow$ No Lagrange multiplier α in derivation

$\Rightarrow$ no μ

Otherwise just like Bose-Einstein distribution

$$n_s = \frac{1}{e^{\beta \varepsilon_s} - 1} \qquad \varepsilon_s = \hbar \omega_s$$

Same result, different ways of thinking about same thing

Typically work with spectral density

$$u(\omega)d\omega = \frac{\text{energy}}{\text{volume}} \text{ with frequency in range } \omega \text{ to } \omega + d\omega$$

Know energy/mode $= \dfrac{\hbar\omega_s}{e^{\beta\hbar\omega_s} - 1}$

Need # of modes/volume ~ density of states

Modes $e^{i\vec{k}\cdot\vec{r}}$ for periodic boundary

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$\Rightarrow (\Delta k)^3 \text{ for one mode} = \frac{(2\pi)^3}{V}$$

# of modes between k & $k+dk$

$$= \frac{4\pi k^2 dk}{\frac{(2\pi)^3}{V}} \times \underset{\substack{\uparrow \\ \text{2 polarizations}}}{2}$$

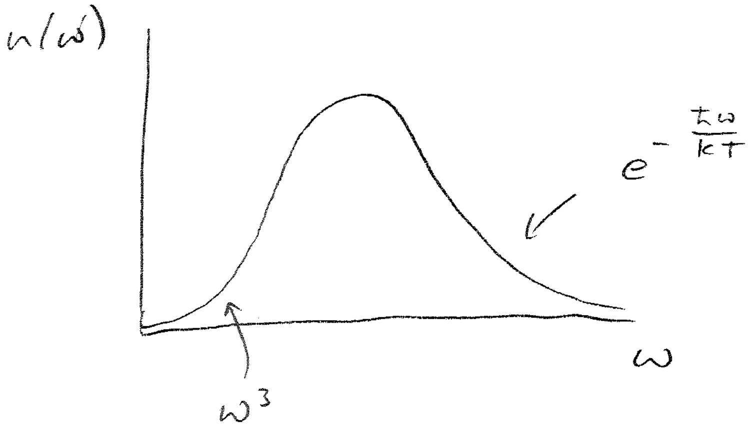
$$k = \frac{\omega}{c}$$

So # of modes between ω & $\omega + d\omega$

$$= V \frac{\omega^2 d\omega}{\pi^2 c^3}$$

and $$u(\omega) = \left(\frac{V\omega^2}{\pi^2 c^3}\right)\left(\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\right) \cdot \frac{1}{V}$$

$$\boxed{u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}}$$ Planck's formula



Note that $\omega = 0$ modes do have $n_0 \to \infty$

But modes have no energy, doesn't matter!

If we have cavity w/ volume V

total energy

$$U = V \int u(\omega)\, d\omega \qquad x = \beta\hbar\omega$$

$$= \frac{\hbar}{\pi^2 c^3} \frac{1}{(\beta\hbar)^4} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\, dx}_{\Gamma(4)\, g_4(1)}$$

Turns out $g_\nu(1)$ can be evaluated in closed form when $\nu =$ even integer

$$g_2(1) = \frac{\pi^2}{6} \qquad g_4(1) = \frac{\pi^4}{90}$$

$$\Gamma(4) = 3! = 6$$

So $$\frac{U}{V} = \frac{\pi^2}{15} \frac{(kT)^4}{(\hbar c)^3} \quad \propto T^4$$

If cavity has small hole in it, radiation leaks out

$$\text{Emittance} = \frac{\text{Power}}{\text{Area of hole}}$$

$$= \frac{u}{V} \times \frac{c}{4}$$

↳ get $\frac{1}{4}$ from angular integrals over photon directions ... see § 6.4

So $$I = \frac{\pi^2}{60} \frac{k^4}{\hbar^3 c^3} T^4 \equiv \sigma T^4$$

Stephen-Boltzmann law

Get same emittance for any perfect radiator at temperature T

Can do thermo as for Bose gas with $z = 1$

$$\ln \mathcal{Z} = \frac{PV}{kT} = - \sum_{\varepsilon} \ln (1 - e^{-\beta \varepsilon})$$

Convert sum to integral using photon density of states

$$a(\omega)\, d\omega = V \frac{\omega^2 d\omega}{\pi^2 c^3}$$

$$a(\varepsilon)\, d\varepsilon = V \frac{\varepsilon^2 d\varepsilon}{\pi^2 \hbar^3 c^3}$$

$$\frac{PV}{kT} = - \frac{V}{\pi^2 (\hbar c)^3} \int_0^\infty \ln (1 - e^{\beta \varepsilon})\, \varepsilon^2 d\omega$$

Integrate by parts

$$\to \frac{V}{3\pi^2(\hbar c)^3}\,\beta \int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta\varepsilon}-1}$$

$$PV = \frac{V}{3\pi^2(\hbar c)^3} \underbrace{\int_0^\infty \frac{x^3 dy}{e^x-1}}_{\pi^4/15}$$

$$PV = \frac{V}{45\pi^2(\hbar c)^3}(kT)^4$$

$$= \frac{1}{3}U$$

General result for extreme relativistic gas.

Can derive in E&M too!

Other thermo follows easily, since $\mu = 0$

$$A = U - TS = -PV + \mu N = -PV = -\frac{1}{3}U$$

$$S = \frac{U-A}{T} = \frac{4}{3}\frac{U}{T} \propto T^3$$

$$C_V = T\frac{\partial S}{\partial T} = 3S$$

etc

But radiation formula is most useful

Similar reasoning applies to other types of waves

Most important: sound waves in solid

= gas of phonons

Recall from mechanics that internal motions of solid reduce to normal modes

= collection of harmonic oscillators

N particles in solid → 3N normal modes

freqs ω_s

Main difference from photons:

Black body radiation has ∞ modes

Sound waves finite number

Matters when $kT \gtrsim \hbar\omega_{max}$

Here write $$U = E_0 + \sum_{s=1}^{3N} \frac{\hbar\omega_s}{e^{\beta\hbar\omega_s} - 1}$$

Negative, = binding energy of crystal

Particularly interested in $C_V = \frac{\partial U}{\partial T}$

Heat capacity of solid

$$C_V = \sum_s \frac{\hbar\omega_s \left(\frac{\hbar\omega_s}{kT^2}\right) e^{\beta\hbar\omega_s}}{(e^{\beta\hbar\omega_s} - 1)^2}$$

$$= k \sum_s \frac{(\beta\hbar\omega_s)^2 e^{\beta\hbar\omega_s}}{(e^{\beta\hbar\omega_s} - 1)^2}$$

To go further, need to know what ω's are

Not really possible to calculate

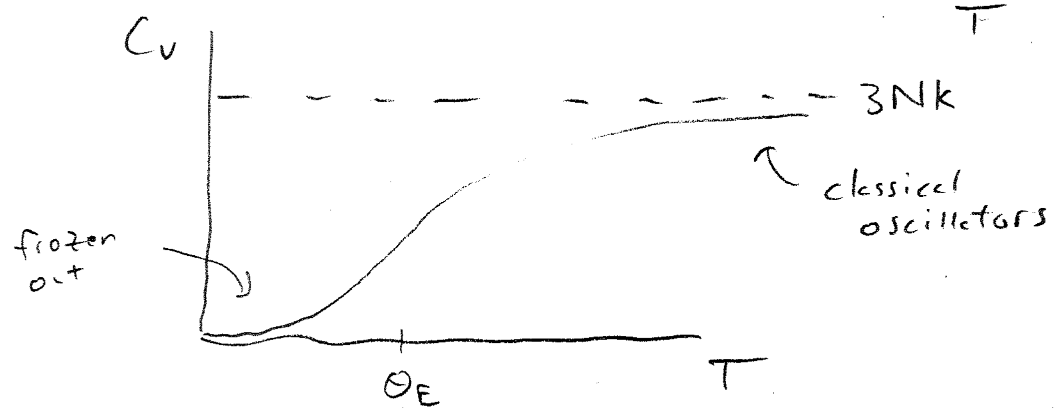
Instead make plausible guesses

Einstein model: all $\omega_s = \omega_E$ constant

$\rightarrow$ treat as non-interacting masses on springs

$$C_V \rightarrow 3Nk \times \frac{x^2 e^x}{(e^x - 1)^2} \qquad x = \beta \hbar \omega_E = \frac{\hbar \omega_E}{kT} \equiv \frac{\Theta_E}{T}$$

Einstein temperature



Agrees qualitatively w/ experiment

Θ_E typically 100's of K

Better agreement with Debye model:

Treat phonons just like photons

$$\omega_s = \frac{\pi c}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} \qquad n's = 0, 1, 2 \ldots$$

But cut off spectrum at max freq ω_D

Also $c \rightarrow$ speed of sound

However, speed of sound is different for longitudinal (compression) waves and transverse (shear) waves

c_L & c_T

So write $\displaystyle\int_0^{\omega_D} \frac{\omega^2 d\omega}{2\pi^2 c_L^3} + \frac{\omega^2 d\omega}{\pi^2 c_T^3} = \int g(\omega)\, d\omega = 3N$

Solve & get $\displaystyle\omega_D^3 = 18\pi^2 \frac{N}{V}\left(\frac{1}{c_L^3} + \frac{2}{c_T^3}\right)^{-1}$

If $g(\omega)$ gives spectral density of modes,

$$g(\omega) = \begin{cases} \dfrac{9N}{\omega_D^3}\,\omega^2 & \omega < \omega_D \\ 0 & \omega > \omega_D \end{cases}$$

Note that real solids have much more complicated g's

Usually take this form only for $\omega \ll \omega_D$

Then get

$$C_V = k \int g(\omega) \frac{(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}\, d\omega$$

$$= \frac{9Nk}{\omega_D^3} \frac{1}{\beta^2\hbar^2} \int_0^{\omega_D} \frac{(\beta\hbar\omega)^4 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}\, d\omega$$

$$= \frac{9Nk}{(\beta\hbar\omega_D)^3} \int_0^{x_0} \frac{x^4 e^x}{(e^x-1)^2}\, dx$$

$x = \beta\hbar\omega$

$x_0 = \beta\hbar\omega_D \equiv \dfrac{\Theta_D}{T}$

Θ_D = Debye temperature

$$C_v = 3Nk\, D(x_0)$$

$$D(x_0) = \frac{3}{x_0^3} \int_0^{x_0} \frac{x^4 e^x}{(e^x - 1)^2}\, dx$$

Debye function

High T limit $x_0 \to 0$

$$D \to \frac{3}{x_0^3} \int_0^{x_0} \frac{x^4}{x^2}\, dx = \frac{3}{x_0^3} \cdot \frac{x_0^3}{3} = 1$$

So $C_v \to 3Nk$, classical result

Low T limit $x_0 \to \infty$

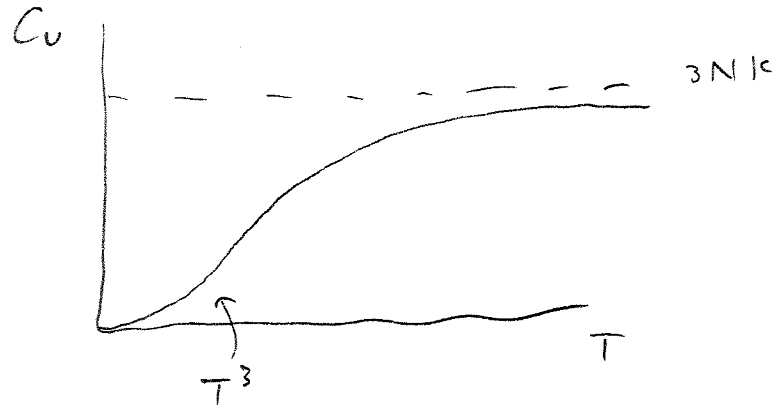
$$D \to \frac{3}{x_0^3} \int_0^{\infty} \frac{x^4 e^x}{(e^x - 1)^2}\, dx$$

Integrate by parts

$$\to \frac{12}{x_0^3} \underbrace{\int_0^{\infty} \frac{x^3}{e^x - 1}\, dx}_{6 g_4(1) = \frac{\pi^4}{15}}$$

$$D \to \frac{4\pi^4}{5 x_0^3} = \frac{4\pi^4}{5} \left(\frac{T}{\Theta_D}\right)^3$$

$$C_v \to \frac{12\pi^4}{5} N k_B \left(\frac{T}{\Theta_D}\right)^3$$



Typically agrees quite well with experiment

Note T^3 vs $e^{-1/T}$ for Einstein

at low T, have low freq modes available

so excitation not completely frozen