Lecture 22

Last time, started discussion of Bose gases

$$n_\varepsilon = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \qquad \text{distribution function}$$

$$N = N_0 + \frac{V}{\Lambda^3} g_{3/2}(z)$$

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x - 1}\, dx$$

$$\Lambda = \left(\frac{h^2}{2\pi m kT}\right)^{1/2}$$

$$z = e^{\beta\mu} \qquad N_0 = \frac{z}{1-z}$$

$$P = \frac{kT}{\Lambda^3} g_{5/2}(z)$$

Can get all thermo properties

One peculiar phenomenon:

Know $\mu < 0 \iff z < 1$

As $z \to 1$, $N_0 \to \infty$

$$\zeta\left(\tfrac{3}{2}\right) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots = 2.612$$

But $N_e \equiv N - N_0 \to \frac{V}{\Lambda^3} g_{3/2}(1) = 2.612 \frac{V}{\Lambda^3}$

finite number of particles possible in excited state

If $N > N_e(\text{max})$, balance of particles $\to$ ground state

Really do get large population in one state:

Bose-Einstein condensation

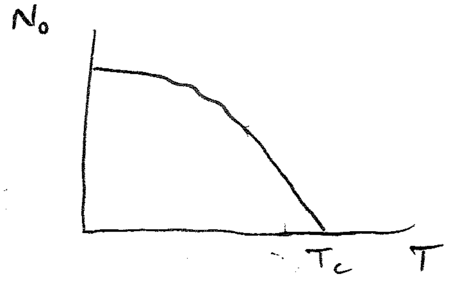
Define $N_c = \zeta(\tfrac{3}{2}) \dfrac{V}{\Lambda^3} = \zeta(\tfrac{3}{2}) V \left(\dfrac{2\pi m k T}{h^2}\right)^{3/2}$

Get condensate when $N > N_c$

Or in terms of T, $T < T_c = \dfrac{h^2}{2\pi m k}\left[\dfrac{N}{V\zeta(\frac{3}{2})}\right]^{2/3}$

Below T_c, get

$$N_0 = N - N_c(T)$$
$$= N\left[1 - \left(T/T_c\right)^{3/2}\right]$$



As $T \to 0$, $N_0 \to N$, but that's not surprising.

Happens anyway for a classical gas

But N_0 gets large at much higher T.

See difference:

Classical $\quad n_\varepsilon = e^{\beta(\mu - \varepsilon)}$

$$N_0^{(cl)} = e^{\beta\mu} = z$$

Boson $\quad N_0^{(B)} = \dfrac{z}{1-z}$

So if $N_0^{(B)} = \dfrac{N}{2} \Rightarrow z = \dfrac{N_0}{N_0+1} = \dfrac{N}{N+2}$,

$N_0^{(c)}$ would $= \dfrac{N}{N+2} \approx 1$

BEC works like a phase transition

Though normal phase transitions require interacting particles

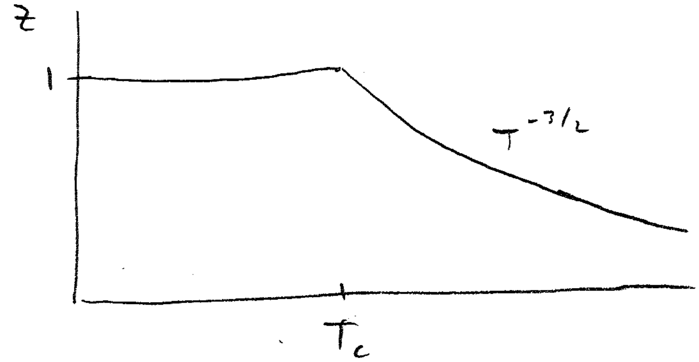
BEC does not

Explore thermodynamics:

z vs T:

For $T<T_c$, $z \approx \frac{N_0}{N_0+1} \approx 1$

For $T>T_c$, $g_{3/2}(z) = \frac{N\Lambda^3}{V} \sim T^{-3/2}$



(fixed N, V)

P vs T:

For $T<T_c$, $P = \frac{kT}{\Lambda^3}\,\zeta(\tfrac{5}{2}) \propto T^{5/2}$

Note P independent of V:

compressibility $\kappa_T = \infty$

At T_c, $P(T_c) = \frac{kT_c}{\Lambda^3}\,\zeta(\tfrac{5}{2})$

but at T_c, $\Lambda^3 = \frac{V}{N}\frac{1}{\zeta(\frac{3}{2})}$

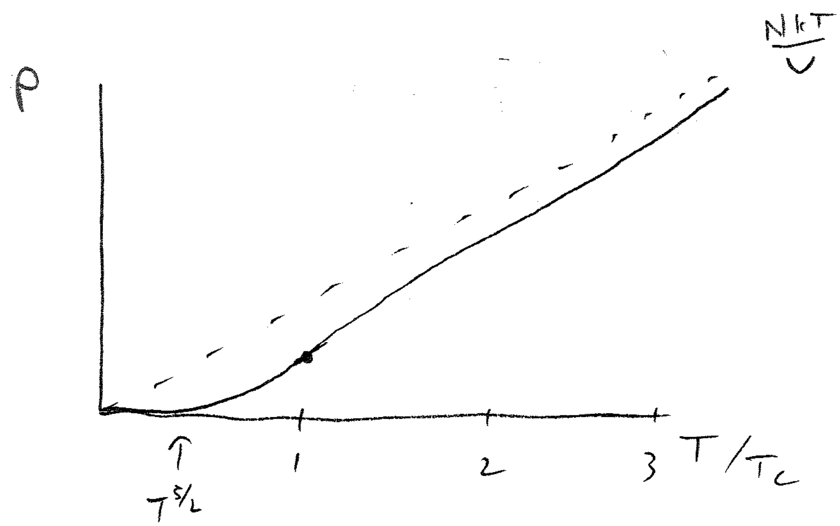
So $P(T_c) = \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(\frac{N}{V}kT_c\right) \approx 0.51\,\frac{NkT_c}{V}$

$\approx \frac{1}{2}P_{classical}$

For $T>T_c$, $P = \frac{NkT}{V} \times \frac{g_{5/2}(z)}{g_{3/2}(z)}$

For small z, $g_\nu(z) \to z$

so $\quad p \to \dfrac{NkT}{V} = p_{classical}$



(fixed N, V)

Last time, derived $\quad U = \frac{3}{2} kT \frac{V}{\lambda^3} g_{5/2}(z)$

$$= \frac{3}{2} pV$$

So U vs T looks just like p vs T

Heat capacity $\quad C_V = \left(\frac{\partial U}{\partial T}\right)_{N,V}$

For $T < T_c$,

$$C_V = \frac{3}{2} kV \zeta(\tfrac{5}{2}) \underbrace{\frac{d}{dT}\left(\frac{T}{\lambda^3}\right)}_{\frac{5}{2}\frac{1}{\lambda^3}}$$

So $\quad \dfrac{C_V}{Nk} = \dfrac{15}{4} \zeta(\tfrac{5}{2}) \dfrac{V}{N\lambda^3} \propto T^{3/2}$

At $T = T_c$, $\quad \dfrac{C_V}{Nk} = \dfrac{15}{4} \dfrac{\zeta(5/2)}{\zeta(3/2)} = 1.925$

actually greater than classical value 1.5!

(See slope of U vs T is larger than classical)

For $T > T_c$,

$$\frac{C_V}{Nk} = \frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}(z)}{g_{3/2}(z)}\right)_{V,N}$$

Evaluate derivative:

From definition of g_ν, can show

$$z \frac{\partial}{\partial z}\left[g_\nu(z)\right] = g_{\nu-1}(z)$$

(prove using integration by parts)

So write

$$\frac{C_V}{Nk} = \frac{3}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{3}{2} T \frac{\partial}{\partial T}\left[\frac{g_{5/2}(z)}{g_{3/2}(z)}\right]$$

[All derivs at fixed N, V]

Use $\displaystyle \frac{\partial}{\partial T} = \frac{\partial z}{\partial T}\frac{\partial}{\partial z}$

Need $\displaystyle \frac{\partial z}{\partial T}$: use trick:

$$g_{3/2}(z) = \frac{N\Lambda^3}{V} \propto T^{-3/2}$$

So $$\frac{\partial}{\partial T} g_{3/2} = -\frac{3}{2T} g_{3/2}(z)$$

But also $$\frac{\partial}{\partial z} g_{3/2} = \frac{1}{z} g_{1/2}$$

$$= \frac{\partial T}{\partial z}\left(\frac{\partial}{\partial T} g_{3/2}\right) = \frac{\partial T}{\partial z} * \left(-\frac{3}{2T} g_{3/2}\right)$$

Solve $$\boxed{\frac{\partial z}{\partial T} = -\frac{3z}{2T}\frac{g_{3/2}}{g_{1/2}}}$$

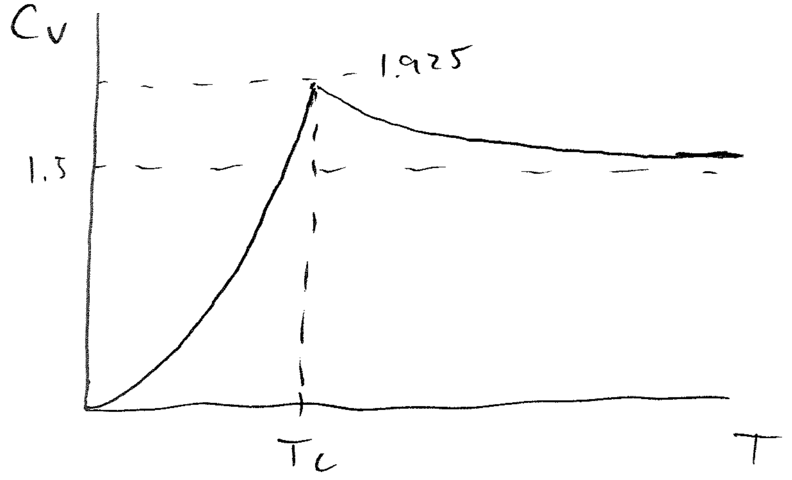
Also need $\dfrac{\partial}{\partial z}\left(\dfrac{g_{5/2}}{g_{7/2}}\right) = \dfrac{1}{z}\dfrac{g_{7/2}}{g_{7/2}} - \dfrac{1}{z}\dfrac{g_{5/2}\,g_{1/2}}{g_{3/2}^2}$

Put together:

$$\frac{C_v}{Nk}(T>T_c) = \frac{3}{2}\frac{g_{5/2}}{g_{7/2}}$$

$$\underbrace{+\frac{3}{2}T\left(-\frac{3z}{2T}\frac{g_{7/2}}{g_{1/2}}\right)\left(\frac{1}{z}-\frac{1}{z}\frac{g_{5/2}\,g_{1/2}}{g_{3/2}^2}\right)}_{-\frac{9}{4}\left(\frac{g_{3/2}}{g_{1/2}}-\frac{g_{5/2}}{g_{3/2}}\right)}$$

$$\boxed{\frac{C_v}{Nk} = \frac{15}{4}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{7/2}}{g_{1/2}}}$$



Discontinuous derivative at T_c

Finally, pretty easy to get

$$S = \frac{1}{T}(U + PV - N\mu)$$

Have U, PV, and $\mu = kT\ln z$

Find $S = \frac{5}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \ln z \qquad T > T_c$

$= \frac{5}{2} \frac{V}{N\Lambda^3} \zeta(\frac{5}{2}) \qquad T < T_c$

Note that $S \to 0$ as $T \to 0$

Agrees w/ 3rd Law of Thermo

Unlike Sackur-Tetrode eqn

That's enough about ideal Bose gas

Turn to close cousin: photon gas

= black body radiation

Two ways to think about:

1) E&M cavity with modes $s = 1, 2, \ldots$

Mode s has frequency ω_s

Acts like quantum harmonic oscillator

$\varepsilon_s = n_s \hbar \omega_s$ (ignore zero point energy)

2) Gas of bosons: photons. Allowed energies $\varepsilon_s = \hbar \omega_s$

$s = 1, 2, \ldots$

Don't really seem the same, but these are physically equivalent

Approach 1:

Collection of distinct oscillators

Solve independently in canonical ensemble

→ Lecture 15, text section 3.8

Neglecting zero point energy, get

Q_s = partition fcn for mode s

$$= \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_s} = \frac{1}{1-e^{-\beta \hbar \omega_s}}$$

$$\langle \varepsilon_s \rangle = -\frac{\partial}{\partial \beta} \ln Q_s = \frac{\hbar \omega_s e^{-\beta \hbar \omega}}{1-e^{-\beta \hbar \omega}}$$

$$= \frac{\hbar \omega_s}{e^{\beta \hbar \omega}-1}$$

and $$\langle n_s \rangle = \frac{\langle \varepsilon_s \rangle}{\hbar \omega_s} = \frac{1}{e^{\beta \hbar \omega_s}-1}$$

Approach 2:

Collection of indistinguishable bosons

Different from regular particles though

because number not conserved.

Photons easily created or destroyed...

no constraint $\sum_s n_s = N$

So no Lagrange multiplier α

in derivation

$\Rightarrow$ no μ

So derivation proceeds like Bose-Einstein distribution, but μ never appears

$\Rightarrow z = 1$

$$\Rightarrow \quad n_\varepsilon = \frac{1}{e^{\beta \varepsilon_s} - 1}$$

same result

Two different ways of thinking about same thing

Most convenient to use spectral energy density

$u(\omega)d\omega$ = energy/volume with frequency in range ω to $\omega + d\omega$

Know energy/mode $= \dfrac{\hbar\omega_s}{e^{\hbar\omega_s/kT} - 1}$

Need # of modes/unit volume ~ density of states:

Modes ~ $e^{i\vec{k}\cdot\vec{r}}$ for periodic boundary

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$\Rightarrow \quad (\Delta k)^3 \text{ for one mode} = \frac{(2\pi)^3}{V}$$

# of modes between k + $k+dk$

$$= \frac{4\pi k^2 dk}{\frac{(2\pi)^3}{V}} \times 2$$

2 polarizations

$k = \omega/c$

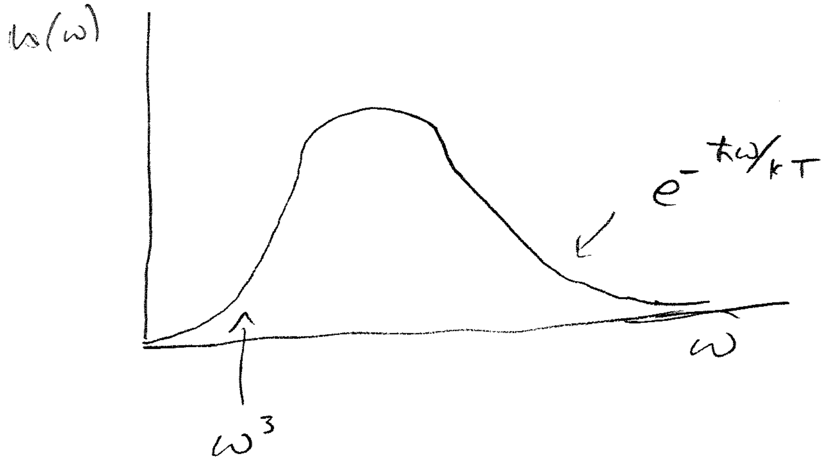
# of modes between ω & $\omega + d\omega$

$$= V \frac{\omega^2 d\omega}{\pi^2 c^3}$$

So $$u(\omega) = \left(\frac{V\omega^2}{\pi^2 c^3}\right) \times \left(\frac{\hbar\omega}{e^{\hbar\omega\beta} - 1}\right) \times \frac{1}{V}$$

$$\boxed{u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}}$$

Planck's formula for spectrum of black body radiation



Note that $\omega = 0$ modes have same problem with $n_0 \to \infty$

But since $\omega = 0$ photons have no energy, this doesn't matter!