Assignment 8

8.1 (Pathria 6.2) (a) Fill in the details for the derivations of Eqs. (6.3.10) and (6.3.11).

(b) Using the probabilities $p_\epsilon(n)$ directly, calculate the variances $\Delta n_\epsilon^2 = \langle n_\epsilon^2\rangle - \langle n_\epsilon\rangle^2$ for both Bose and Fermi systems. Verify Eq. (6.3.9).

(c) For both types of particle, show that

$$\Delta n_\epsilon^2 = kT\left(\frac{\partial \langle n_\epsilon\rangle}{\partial \mu}\right)_T$$

Note the similarity to Eq. (4.5.3).

8.2 (Pathria 6.3) Refer to Sec. 6.2 and show that, if the occupation number n_ϵ of an energy level ϵ is restricted to the values $0, 1, \ldots, \ell$, then the mean occupation number of that level is given by

$$\langle n_\epsilon\rangle = \frac{1}{z^{-1}e^{\beta\epsilon} - 1} - \frac{\ell+1}{(z^{-1}e^{\beta\epsilon})^{\ell+1} - 1}$$

Check that $\ell = 1$ leads to the Fermi-Dirac distribution and $\ell \to \infty$ leads to the Bose-Einstein distribution.

8.3 (Pathria 6.10) (a) Show that the momentum distribution of particles in a relativistic Boltzmann gas, with $\epsilon = c(p^2 + m^2c^2)^{1/2}$, is given by

$$f(\mathbf{p})d^3p = C\exp\left[-\beta c\left(p^2 + m^2c^2\right)^{1/2}\right]p^2dp,$$

with the normalization constant

$$C = \frac{\beta}{m^2cK_2(\beta mc^2)},$$

where $K_\nu(z)$ is the modified Bessel function, with integral representation

$$K_\nu(z) = \frac{\sqrt{\pi}(z/2)^\nu}{\Gamma(\nu+1/2)}\int_1^\infty e^{-zt}\left(t^2-1\right)^{\nu-1/2}dt.$$

(b) Check that in the nonrelativistic limit ($kT \ll mc^2$) we recover the Maxwell distribution

$$f(\mathbf{p})d^3p = \left(\frac{\beta}{2\pi m}\right)^{3/2}\exp\left(-\frac{\beta p^2}{2m}\right)\left(4\pi p^2dp\right).$$

(c) Verify explicity that in all cases, $\langle pu\rangle = 3kT$, as seen already in problem 6.3 (Pathria 3.24).

8.4 (Pathria 7.2) Deduce the virial expansion (7.1.13) from Eqs. (7.1.7) and (7.1.8), and verify the quoted values of the virial coefficients in (7.1.14) up to third-order. (You don't need to do the a_4 term.)