Assignment 7

7.1 (Pathria 4.4.) (a) Show that the probability for a system in the grand canonical ensemble to have exactly N particles is

$$p(N) = \frac{z^N Q_N(V,T)}{\mathcal{Q}(z,V,T)}.$$

(b) The Poisson distribution is defined by

$$p(N) = e^{-\bar{N}} \frac{\bar{N}^N}{N!}$$

where $\bar{N}$ is the average $\langle N \rangle$. Show that for a classical ideal gas, the probability distribution defined in (a) corresponds to a Poisson distribution.

(c) For an ideal gas, calculate the variance in particle number ΔN^2 using both the general formula (4.5.3) and by direct calculation from the Poisson distribution. Verify that the results are identical.

7.2 (Pathria 4.6) Define the function $Y(N, \gamma, T)$ by

$$Y = \int_0^\infty Q_N(V,T) e^{-\gamma V} dV.$$

This can be taken as the normalization function for an ensemble of systems with different V, suggesting that it could describe an ensemble with constant pressure P. Determine the relation between γ and P, and between Y and the Gibbs free energy $G(N,P,T)$. Verify explicitly that your relations are correct for an ideal gas. You may assume here that N is very large.

7.3 (Pathria 5.3) In section 5.3, the text analyzes the density matrix for a harmonic oscillator, using the position basis. Repeat this calculation using the basis of momentum eigenstates. Calculate the density matrix $\langle p|\rho|p'\rangle$, the momentum density $\langle p|\rho|p\rangle$ and the root mean square deviation of the momentum, all as functions of β. (A good approach to this problem is to make use of the near symmetry between p and q for a harmonic oscillator; this makes much of the math identical to that presented in the text.)

7.4 (Pathria 5.6) Calculate the degeneracy factor $n\Lambda^3$ for gases of hydrogen, helium, and oxygen at standard temperature and pressure (300 K and 1 atmosphere). Estimate the temperature at which quantum effects would become important, if the density is held fixed at the standard value.

7.5 The density matrix for the spin of an electron is given, in the $\{\uparrow_z, \downarrow_z\}$ basis, by

$$\begin{bmatrix} a & ce^{i\phi} \\ ce^{-i\phi} & b \end{bmatrix}$$

with $a + b = 1$ and c real. Calculate the probabilities for the spin to be found in each of the states $\uparrow_z$, $\downarrow_z$, $\uparrow_x$, $\downarrow_x$ $\uparrow_y$, and $\downarrow_y$, where $\uparrow_i$ indicates a projection of angular momentum $+\hbar/2$ along axis i, and $\downarrow_i$ represents a projection $-\hbar/2$. For reference, the spin states are related according to

$$\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z) \qquad \downarrow_x = \frac{1}{\sqrt{2}}(\uparrow_z - \downarrow_z)$$

$$\uparrow_y = \frac{1}{\sqrt{2}}(\uparrow_z + i \downarrow_z) \qquad \downarrow_y = \frac{1}{\sqrt{2}}(\uparrow_z - i \downarrow_z)$$

(See for instance Griffiths, Ch 4.4.)