Assignment 5

These problems are from Pathria, but I've reworded them to be (I hope) a little clearer.

5.1 (Pathria 2.4.) A classical rigid rotator is a simple model for a diatomic molecule, consisting of two point masses separated by a fixed distance. The state of the rotator is determined by the orientation of its axis, with spherical coordinates (θ, ϕ) and corresponding momenta (p_θ, p_ϕ). Classical mechanics gives us the angular momentum M of the rotator as

$$M^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$$

(a) Evaluate the volume of phase space available to a rigid rotator with $M \le M_0$, and show that the corresponding number of microstates available is $(M_0/\hbar)^2$.

(b) Quantum mechanics tells us that angular momentum is quantized as $M_j = \hbar\sqrt{j(j+1)}$ with either $j = 0, 1, 2, ...$ or $j = 1/2, 3/2, 5/2, ...$, in either case with a degeneracy $2j+1$. Use this to evaluate the number of states available to a quantum rotator with $j \le j_0$, and compare to the result of (a).

5.2 (Pathria 2.8) (a) Particles moving at relativistic speeds are characterized by an energy-momentum relation $\epsilon = |\mathbf{p}|c$. Set up an integral to determine the volume of phase space available to an ensemble with N relativistic particles with energy less than or equal to E.

(b) To evaluate the momentum integrals, prove that

$$I_{3N} \equiv \int \cdots \int_{0 \le \Sigma_{i=1}^N r_i \le R} \prod_{i=1}^{N} (4\pi r_i^2 dr_i) = \frac{(8\pi R^3)^N}{(3N)!}.$$

This can be achieved using the method of Appendix C, replacing Eq. (C.4) by the identity

$$\int_0^\infty e^{-r} r^2 dr = 2$$

(c) Using the results of (a) and (b), calculate the entropy of a relativistic ideal gas. Show that the ratio $C_P/C_V = 4/3$.

5.3 (Pathria 2.9) Consider now a gas of $3N$ relativistic particles moving in one dimension, confined to a length L.

(a) Compute the phase space volume available for energy less than or equal to E. The integral you get is related to that obtained in problem 4.2; see the solutions to assignment 4. Calculate the entropy of the gas.

(b) The quantum mechanical eigenstates for this system have energies $(hc/2L)n$ for $n = 1, 2, \ldots$. In this case, it is possible to exactly calculate the number of ways to distribute the energy E over the system of N particles, using combinatorics methods. (The problem is equivalent to distributing E^* marbles into N different boxes, for an appropriate E^*.) Using this approach, calculate the entropy of the gas using $S = k_B \log\Omega$, and show that it converges to the result obtained in (a) in the limit of large E/N.

(c) Compare the thermodynamics of this system to that considered in problem 5.2.