1.

a) Have, in general,

$$N = \sum_{\varepsilon} \frac{1}{z^{-1}e^{\beta\varepsilon} - 1}, \quad \text{sum over all states } \varepsilon$$

Here have $\frac{n^2}{2}$ states at energy $\varepsilon_n = \hbar\omega(n+\frac{3}{2})$

So

$$N = \sum_{n=0}^{\infty} \frac{n^2}{2} \frac{1}{z^{-1}e^{\beta\hbar\omega(n+\frac{3}{2})} - 1}$$

Given $kT \gg \hbar\omega$, so $\beta\hbar\omega \ll 1$

$$\Rightarrow \frac{3}{2}\beta\hbar\omega \ll 1$$

Could just neglect this term

Otherwise, incorporate into z:

$$\bar{z}^{-1} = z^{-1}e^{\frac{3}{2}\beta\hbar\omega}$$

$$N = \sum_{n} \frac{n^2}{2} \frac{1}{\bar{z}^{-1}e^{\beta\hbar\omega n} - 1}$$

Convert to integral. Already have $\Delta n = 1$, so

$$\sum \Delta n \to \int dn$$

First, pull out ground state

$$N = \frac{1}{\bar{z}^{-1} - 1} + \frac{1}{2}\int_0^{\infty} \frac{n^2\,dn}{\bar{z}^{-1}e^{\beta\hbar\omega n} - 1}$$

See that $N_0 = \frac{1}{\bar{z}^{-1} - 1} \to \infty$ as $\bar{z} \to 1$

So get N_c = max # in excited state

$$= \frac{1}{2}\int_0^\infty \frac{n^2\,dn}{e^{\beta\hbar\omega n}-1} \qquad x = \beta\hbar\omega n$$

$$= \frac{1}{2}(\beta\hbar\omega)^{-3}\int_0^\infty \frac{x^2\,dx}{e^x-1}$$

$$= \frac{1}{2}\left(\frac{kT}{\hbar\omega}\right)^3 \Gamma(3)\, g_3(1)$$

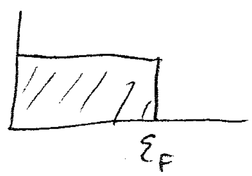
$$\Gamma(3) = 2! = 2$$

$$g_3(1) = \zeta(3) = 1.202$$

$$\boxed{N_c = 1.202\left(\frac{kT}{\hbar\omega}\right)^3}$$

b) Get Fermi energy as $T \to 0$ or $\beta \to \infty$

$f(\varepsilon) \to$



Then
$$N = \sum_\varepsilon \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \to \sum_{\varepsilon<\varepsilon_F} 1$$

$$= \sum_{n<n_F} \frac{n^2}{2} \qquad \text{for} \quad \varepsilon_F = \hbar\omega\left(n_F + \tfrac{3}{2}\right) \approx n_F\hbar\omega$$

approximate as integral:
$$N = \frac{1}{2}\int_0^{n_F} n^2\,dn = \frac{n_F^3}{6}$$

$$n_F = (6N)^{1/3} \Rightarrow \boxed{\varepsilon_F = (6N)^{1/3}\hbar\omega}$$

2. $$Q_1 = \sum_{m=-1}^{1} e^{-\beta\varepsilon_m} \qquad \varepsilon_m = -m\mu^*\mathcal{H}$$

$$Q_1 = e^{-\beta\mu^*\mathcal{H}} + 1 + e^{\beta\mu^*\mathcal{H}}$$

$$= 1 + 2\cosh x \qquad \boxed{x \equiv \beta\mu^*\mathcal{H}}$$

Then $$M = \frac{N}{V}\langle m\mu^*\rangle$$

$$= \frac{N}{V}\sum \frac{m\mu^* e^{\beta m\mu^*\mathcal{H}}}{Q_1}$$

$$= n\mu^* \frac{e^x - e^{-x}}{Q_1}$$

$$M = n\mu^* \frac{2\sinh x}{1 + 2\cosh x}$$

$$\chi = \frac{\partial M}{\partial \mathcal{H}} = n\mu^* \frac{\partial x}{\partial \mathcal{H}}\left[\frac{2\cosh x}{1+2\cosh x} - \frac{(2\sinh x)(2\sinh x)}{(1+2\cosh x)^2}\right]$$

$$= 2n\mu^{*2}\beta \frac{\cosh x(1+2\cosh x) - 2\sinh^2 x}{(1+2\cosh x)^2}$$

Note $\cosh^2 x - \sinh^2 x = 1$

$$\boxed{\chi = 2n\mu^*\beta \frac{\cosh x + 2}{(1+2\cosh x)^2}}$$

3.

a) $$Q_1 = \frac{1}{h^3} \int e^{-\beta H} d^3q\, d^3p$$

Here $H = \frac{p^2}{2m} + mgz$

$$Q_1 = \frac{1}{h^3} \int dx\,dy \int_0^L e^{-\beta mgz} dz \int_0^\infty e^{-\beta \frac{p^2}{2m}} 4\pi p^2 dp$$

$$= \frac{1}{h^3} [A] \left[\frac{1}{\beta mg} \left(1 - e^{-\beta mgL}\right) \right] \left(\frac{2\pi m}{\beta} \right)^{3/2}$$

$$Q_1 = \frac{A}{\Lambda^3} \frac{1}{\beta mg} \left(1 - e^{-\beta mgL}\right) \qquad \Lambda^3 = \left(\frac{h^2 \beta}{2\pi m} \right)^{3/2}$$

$$\frac{U}{N} = -\frac{\partial}{\partial \beta} \ln Q_1$$

$$= -\frac{\partial}{\partial \beta} \left[-\frac{5}{2} \ln \beta + \ln\left(1 - e^{-\beta mgL}\right) \right]$$

$$= \frac{5}{2} \frac{1}{\beta} - \frac{mgL\, e^{-\beta mgL}}{1 - e^{-\beta mgL}}$$

$$= \frac{5}{2} kT - \frac{mgL}{e^{\beta mgL} - 1}$$

$$C_V = \frac{\partial U}{\partial T} = \frac{5}{2} Nk - NmgL \left(\frac{\partial \beta}{\partial T} \right) \left(\frac{\partial}{\partial \beta} \frac{1}{e^{\beta mgL} - 1} \right)$$

$$- NmgL \left(-\frac{1}{kT^2} \right) \left(\frac{-mgL\, e^{\beta mgL}}{(e^{\beta mgL} - 1)^2} \right)$$

$$= Nk \left[\frac{5}{2} - \left(\frac{mgL}{kT} \right)^2 \frac{-1}{\left(e^{\beta mgL/2} - e^{-\beta mgL/2}\right)^2} \right]$$

$$\boxed{C_V = Nk\left[\frac{5}{2} - \frac{x^2}{\sinh^2 x}\right] \qquad x = \frac{mgL}{2kT}}$$

b) Need to show $\frac{x^2}{\sinh^2 x} < 1$

$$x^2 < \sinh^2 x$$

$$x < \sinh x \qquad (\text{have } x > 0)$$

But $\sinh x = x + \frac{x^3}{6} + \frac{x^5}{5!} + \ldots$

which is indeed greater than x

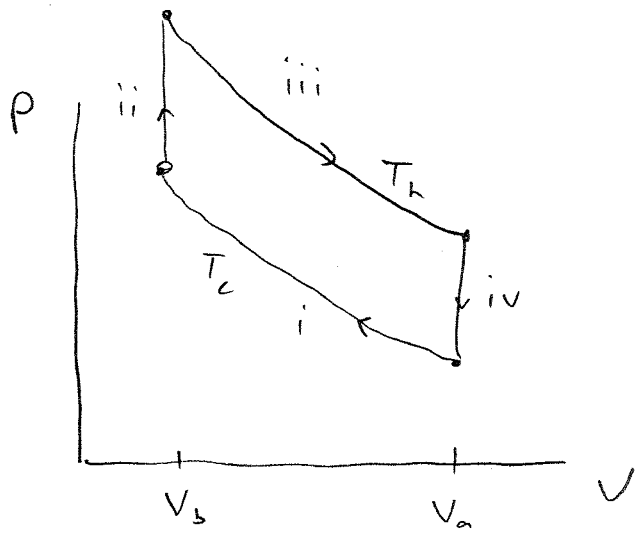
So $C_V > \frac{3}{2} NkT$

Why? Can put energy into gravitational potential... gives one more degree of freedom

$\Rightarrow$ can absorb more heat for a given temperature increase.

4.

a) Picture:



Eqn for isotherm is $P = \frac{NkT}{V}$

Work $\quad W = \oint P\,dV$

$$= \int_{V_b}^{V_a} \frac{NkT_h}{V}\,dV - \int_{V_b}^{V_a} \frac{NkT_c}{V}\,dV$$

$$= Nk(T_h - T_c)\ln V_a/V_b$$

Take up heat

$$Q_{ii} = C_V(T_h - T_c) = \alpha Nk(T_h - T_c)$$

and $\quad Q_{iii} = \Delta U + W_{iii}$

But in step (iii), $\Delta U = 0$ since $T = \text{const}$:

so $\quad Q_{iii} = W_{iii} = NkT_h \ln V_a/V_b$

Total heat

$$Q = \alpha Nk(T_h - T_c) + NkT_h \ln V_a/V_b$$

$$\boxed{\eta = \frac{W}{Q} = \frac{(T_h - T_c)\ln V_a/V_b}{\alpha(T_h - T_c) + T_h \ln V_a/V_b}}$$

b) For Carnot cycle, $\eta_c = 1 - \frac{T_c}{T_h} = \frac{T_h - T_c}{T_h}$

So $$\eta_s = \frac{T_h \eta_c \ln V_a/V_b}{\alpha T_h \eta_c + T_h \ln V_a/V_b}$$

$$= \eta_c \times \frac{\ln V_a/V_b}{\alpha \eta_c + \ln V_a/V_b}$$

Since $\alpha \eta_c > 0$, have $\boxed{\eta_s < \eta_c}$

Why? Because in Stirling cycle, not all heat is taken up at T_h ... heat taken up at low T is less efficient for producing work, as we know from Carnot engine.

5.

$$\ln \mathcal{Q} = \sum_{\varepsilon} \ln\left(1 + z e^{-\beta\varepsilon}\right)$$

$$= -\Phi = \frac{PV}{kT}$$

Convert to integral: $\varepsilon = p^2/2m$

$$\sum_{\varepsilon} \to g \int \frac{dp\,dq}{h}$$

$g = 2J+1$ degeneracy

$$\ln \mathcal{Q} \to \frac{gL}{h} \int_{-\infty}^{\infty} \ln\left(1 + z e^{-\beta p^2/2m}\right) dp$$

$$= \frac{2gL}{h} \int_0^{\infty} \ln\left(1 + z e^{-\beta p^2/2m}\right) dp$$

$x = \beta \frac{p^2}{2m}$

$\Rightarrow dp = \sqrt{\frac{m}{2\beta x}}\, dx$

$$\ln \mathcal{Q} = \frac{2gL}{h}\sqrt{\frac{m}{2\beta}} \int_0^{\infty} x^{-1/2} \ln\left(1 + z e^{-x}\right) dx$$

Integrate by parts: $u = \ln(1 + z e^{-x})$ $\quad dv = x^{-1/2}$

$du = \frac{-z e^{-x}}{1 + z e^{-x}}$ $\quad v = 2x^{1/2}$

$= \frac{-1}{z^{-1} e^{x} + 1}$

$uv\Big|_0^{\infty} = 0$, so

$$\ln \mathcal{Q} = gL\sqrt{\frac{2m}{\beta h^2}} \int_0^{\infty} \frac{2x^{1/2}\, dx}{z^{-1} e^{x} + 1}$$

Use : $\int_0^\infty \frac{x^{1/2}}{z^{-1}e^x+1}\,dx = \Gamma(\tfrac{3}{2})\, f_{3/2}(x)$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

So $$\ln Q = 2gL\sqrt{\frac{2m}{\beta h^2}}\;\frac{\sqrt{\pi}}{2}\, f_{3/2}(z)$$

$$= gL\sqrt{\frac{2\pi m kT}{h^2}}\; f_{3/2}(z)$$

$$= \frac{PV}{kT} \rightarrow \frac{PL}{kT}$$

and $$\boxed{P = (2J+1)\,\frac{kT}{\Lambda}\, f_{3/2}(z)}$$

$$\Lambda = \sqrt{\frac{h^2}{2\pi m kT}}$$

Here $U = 0$ always, and get

$$J = -T \frac{\partial S}{\partial L} = -kT \left[-\frac{\partial N_r}{\partial L} \frac{\partial}{\partial N_r} \left(N_r \ln N_r \right) - \frac{\partial N_e}{\partial L} \frac{\partial}{\partial N_e} \left(N_e \ln N_e \right) \right]$$

$$= kT \left[\left(\frac{1}{2d} \right) \left(\ln N_r + 1 \right) \left(-\frac{1}{2d} \right) \left(\ln N_e + 1 \right) \right]$$

$$= \frac{kT}{2d} \left[\ln N_r - \ln N_e \right]$$

$$\boxed{J = \frac{kT}{2d} \left[\ln \tfrac{1}{2} \left(N + \tfrac{L}{d} \right) - \ln \tfrac{1}{2} \left(N - \tfrac{L}{d} \right) \right]}$$

c) For $\frac{L}{d} \ll N$, $\ln \frac{1}{2} \left(N \pm \frac{L}{d} \right) = \ln \frac{N}{2} + \ln \left(1 \pm \frac{L}{Nd} \right)$

$$\rightarrow \ln \frac{N}{2} \pm \frac{L}{Nd}$$

$$J \rightarrow \frac{kT}{2d} \left[\ln \frac{N}{2} + \frac{L}{Nd} - \ln \frac{N}{2} + \frac{L}{Nd} \right]$$

$$\boxed{J = \frac{kT}{Nd^2} L = KL \quad \text{for } K = \frac{kT}{Nd^2}}$$

6.

a) Have $N_r - N_\ell = \frac{L}{d}$

and $N_r + N_\ell = N$

Then
$$\left.\begin{aligned} N_r &= \tfrac{1}{2}\left(N + \tfrac{L}{d}\right) \\ N_\ell &= \tfrac{1}{2}\left(N - \tfrac{L}{d}\right) \end{aligned}\right\} \text{ fixed by } L$$

So # of microstates = # of ways to pick N_r out of N

$$\Omega = \binom{N}{N_r} = \frac{N!}{N_r!\,N_\ell!}$$

and
$$S = k \ln \Omega$$
$$= k\left[N\ln N - N + N_r \ln N_r + N_r - N_\ell \ln N_\ell + N_\ell\right]$$
$$S = k\left[N\ln N - N_r \ln N_r - N_\ell \ln N_\ell\right]$$

$$\boxed{S = k\left[N\ln N - \tfrac{1}{2}\left(N+\tfrac{L}{d}\right)\ln\tfrac{1}{2}\left(N+\tfrac{L}{d}\right) - \tfrac{1}{2}\left(N-\tfrac{L}{d}\right)\ln\tfrac{1}{2}\left(N-\tfrac{L}{d}\right)\right]}$$

b) Know $dU = dQ - dW$
$$= T\,dS + J\,dL$$

So $J = -T\left(\frac{\partial S}{\partial L}\right)_U$