Phys 831 Final Exam 30 April 2009

This is a closed book, closed notes exam, to be taken in a single 3-hour period. The problems should be worked on separate pages and attached to this sheet when completed. There are six problems, which will be weighted equally. For full credit, be sure to show and explain all your work.

Name: ______________________________

Signature: ______________________________

Some values of the Riemann zeta function:

ν	$\zeta(\nu)$
1	∞
3/2	2.612
2	1.645
5/2	1.341
3	1.202

1. A three-dimensional harmonic potential $V = m\omega^2 r^2/2$ supports single-particle eigenenergies $\epsilon_n = \hbar\omega(n + 3/2)$, with $n = 0, 1, 2, \ldots$ The degeneracy of level n approaches $n^2/2$ for $n \gg 1$ (though of course the degeneracy of the ground state is 1). Suppose a gas of identical atoms is confined in such a potential, all with the same (fixed) spin state.

(a) Taking the the atoms to be bosons, calculate the critical atom number N_c for Bose-Einstein condensation as a function of temperature T. (You can assume $kT \gg \hbar\omega$.)

(b) Taking the atoms to be fermions, calculate the Fermi energy as a function of atom number N. (You can assume $N \gg 1$.)

2. Calculate the magnetic susceptibility as a function of field $\mathcal{H}$, for a classical gas of spin-1 particles at temperature T. Take the magnetic moment of a single particle in state $m_J = 1$ to be μ^*.

(To receive full credit, you need to give a complete calculation, not just supply a memorized answer.)

3. A classical ideal gas composed of non-relativistic, spinless atoms is held in a cylindrical column of area A and height L in a uniform gravitational field g. The mass of the atoms is m.

(a) Calculate the heat capacity C_V of the gas in the column.

(b) Show that C_V is larger than $3Nk_B/2$ and explain physically why that is so.

4. The Stirling cycle consists of the following steps:

(i) Isothermal compression from volume V_a to volume V_b at temperature T_c.

(ii) Heating from temperature T_c to T_h at fixed volume V_b.

(iii) Isothermal expansion from V_b to V_a at temperature T_h.

(iv) Cooling back to temperature T_c at fixed volume V_a.

(a) If the Stirling cycle is implemented using an ideal gas having heat capacity $C_V = \alpha N k_B$, calculate the efficiency η, defined here as the work done divided by the total heat intake in steps (ii) and (iii).

(b) Show that the efficiency of the Stirling engine is lower than that of a Carnot engine operating between temperatures T_h and T_c. Explain why that should be the case, given that both engines are reversible.

5. A one-dimensional, non-relativistic Fermi gas is confined in "box" of length L. The mass of the particles is m, and their spin is J. Starting from the grand partition function

$$\ln \mathcal{Q} = \sum_{\epsilon} \ln(1 + z e^{-\beta\epsilon}),$$

calculate the pressure P in terms of the temperature T and chemical potential μ (or equivalently, β and z). Express your answer in terms of the Fermi functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1} e^x + 1}$$

6. A very simple model for an elastic substance like rubber consists of a one-dimensional chain composed of N microscopic links of length d. Each link is attached to the previous link, but can be oriented either to the left or to the right of the attachment point. Both orientations have the same energy, and there is no problem with having several overlapping links in the same place. The total extension of the chain is therefore $L = d(N_r - N_\ell)$, where N_r is the number of links pointing to the right (the direction of positive L) and N_ℓ is the number of links pointing to the left (the direction of negative L).

(a) Calculate the number of microstates corresponding to a given length L, and thus obtain the entropy $S(L)$.

(b) Explain why the tension in the chain is given by $J = -T(\partial S/\partial L)_U$ for temperature T and internal energy U. (You can assume here that $L > 0$ for simplicity.) Noting that U here is anyways constant, evaluate $J(L)$.

(c) Show that for small L, the chain satisfies Hooke's Law $J = KL$ and calculate the spring constant K.