Calculation of Kepler Orbit

Have integral form

$$\phi = \phi_0 \mp \int du \left[\frac{2\mu E}{\ell^2} + \frac{2\mu^2 \gamma}{\ell^2} u - u^2 \right]^{-1/2}$$

Need

$$\int \frac{dx}{\sqrt{Ax^2 + Bx + C}} \qquad A = -1$$

From Dwight, Tables of Integrals and Other Mathematical Data,

have

$$\int \frac{dx}{\sqrt{Ax^2+Bx+C}} = \frac{-1}{\sqrt{-A}} \sin^{-1} \frac{2Ax+B}{\sqrt{B^2-4AC}}$$

$$B^2 - 4AC = \frac{4\mu^4\gamma^2}{\ell^4} + \frac{8\mu E}{\ell^2}$$

$$\sqrt{B^2-4AC} = \frac{2\mu^2\gamma}{\ell^2}\left(1 + \frac{2\ell^2 E}{\mu^3\gamma^2}\right)^{1/2}$$

$$\equiv 2K\varepsilon$$

So

$$\phi = \phi_0 \pm \sin^{-1} \frac{-2u + 2K}{2K\varepsilon}$$

$$\sin^{-1} \frac{K-u}{K\varepsilon} = \mp(\phi - \phi_0)$$

$$\frac{K-u}{K\varepsilon} = \mp \sin(\phi - \phi_0)$$

$$K - u = \mp K\varepsilon \sin(\phi - \phi_0)$$

$$u = K(1 \pm \varepsilon \sin(\phi - \phi_0))$$

Can eliminate $\pm$ through choice of ϕ_0

If "+," define $\phi_0 = \frac{\pi}{2} - \bar{\phi}_0$

$$\sin(\phi - \phi_0) = \sin(\phi - \bar{\phi}_0 - \tfrac{\pi}{2}) = -\cos(\phi - \bar{\phi}_0)$$

If "−" define $\phi_0 = -\frac{\pi}{2} - \bar{\phi}_0$

$$\sin(\phi - \phi_0) = \sin(\phi - \bar{\phi}_0 + \tfrac{\pi}{2}) = +\cos(\phi - \bar{\phi}_0)$$

Thus obtain general form

$$u = K\left[1 - \varepsilon\cos(\phi - \phi_0)\right] \quad \text{as desired}$$

Other integral tables may have different form for solution, so more manipulation may be required to get to standard form.

To derive it for yourself, use:

$$-x^2 + Bx + C = -\left(x - \tfrac{B}{2}\right)^2 + C + \tfrac{B^2}{4}$$

Set $y = x - \frac{B}{2}$, to get

$$\int \frac{dy}{\sqrt{\alpha^2 - y^2}} \qquad \alpha = \sqrt{\tfrac{B^2}{4} + C}$$

$$= \sin^{-1}\frac{y}{\alpha} = \sin^{-1}\frac{2x - B}{\sqrt{B^2 + 4C}},$$ as claimed