Lagrange Multipliers

Want to minimize $\phi(x,y)$
subject to constraint $f(x,y)=0$

Claim that we can obtain this by minimizing

$$\mathcal{H}(x,y,\lambda) = \phi(x,y) - \lambda f(x,y)$$

w/respect to x, y, and λ

Proof:
In principle, can solve $f(x,y)=0$
to get $y = y(x)$

Then $\phi = \phi(x, y(x))$

Minimized when

(1) $$\frac{d\phi}{dx} = \left(\frac{\partial \phi}{\partial x}\right)_y + \left(\frac{\partial \phi}{\partial y}\right)_x \frac{dy}{dx} = 0$$

But this is impractical when $f(x,y)$ is hard to solve

Compare to what we get from $\mathcal{H}(x,y,\lambda)$:

(2) $$\frac{\partial \mathcal{H}}{\partial x} = \left(\frac{\partial \phi}{\partial x}\right)_y - \lambda \frac{\partial f}{\partial x} = 0$$

(3) $$\frac{\partial \mathcal{H}}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_x - \lambda \frac{\partial f}{\partial y} = 0$$

(4) $$\frac{\partial \mathcal{H}}{\partial \lambda} = f(x,y) = 0$$

From (3), $$\lambda = \frac{(\partial \phi/\partial y)_x}{\partial f/\partial y}$$

Substitute into (1):

$$(5) \quad \left(\frac{\partial \phi}{\partial x}\right)_y - \left(\frac{\partial \phi}{\partial y}\right)_x \frac{\partial f/\partial x}{\partial f/\partial y} = 0$$

Also have $f(x,y) = 0$ from (4)

Differentiate implicitly with respect to x:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

so
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Substitute into (5):

$$\left(\frac{\partial \phi}{\partial x}\right)_y + \left(\frac{\partial \phi}{\partial y}\right)_x \frac{dy}{dx} = 0$$

This is the same as (1)

Since we know (1) gives the solution, we conclude that the Lagrange multiplier method does also.

$\square$