1. Since $\vec{L} =$ constant, particle moves in x-y plane

$$(\vec{r} \cdot \vec{L} = \vec{r} \cdot (\vec{r} \times \vec{p}) = 0)$$

Energy $E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r)$

$$l = m r^2 \dot{\phi} \Rightarrow r^2 \dot{\phi}^2 = \frac{l^2}{m^2 r^2}$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2 m r^2} + V(r)$$

$$\dot{r}^2 = \frac{2}{m} \left[E - \frac{l^2}{2 m r^2} - V(r) \right]$$

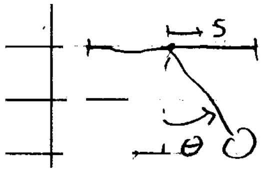
$$\frac{dr}{dt} = \sqrt{\frac{2}{m}} \left[E - \frac{l^2}{2 m r^2} - V(r) \right]^{1/2}$$

$$\frac{dr}{d\phi} = \frac{dr}{dt} \frac{dt}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{m r^2}{l} \sqrt{\frac{2}{m}} \left[E - \frac{l^2}{2 m r^2} - V \right]^{1/2}$$

$$\frac{l}{\sqrt{2m}} \frac{dr}{r^2 \sqrt{E - \frac{l^2}{2 m r^2} - V}} = d\phi$$

$$\boxed{\phi(r) = \frac{l}{\sqrt{2m}} \int \frac{dr}{r^2 \sqrt{E - \frac{l^2}{2 m r^2} - V}} + \phi_0}$$

2. s = position of support
θ = angle of pendulum



$$x = s + l\sin\theta$$
$$y = -l\cos\theta$$

$$\dot{x} = \dot{s} + l\dot{\theta}\cos\theta$$
$$\dot{y} = l\dot{\theta}\sin\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{s}^2 + 2l\dot{s}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta$$
$$= \dot{s}^2 + 2l\dot{s}\dot{\theta}\cos\theta + l^2\dot{\theta}^2$$

So $T = \frac{1}{2}m\left(\dot{s}^2 + 2l\dot{s}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)$

$$V = -mgl\cos\theta + 2\times\left(\tfrac{1}{2}ks^2\right)$$

$$L = \frac{1}{2}m\left(\dot{s}^2 + 2l\dot{s}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta - ks^2$$

Expand to 2nd order:

$$L = \frac{1}{2}m\left(\dot{s}^2 + 2l\dot{s}\dot{\theta} + l^2\dot{\theta}^2\right) + mgl\left(1 - \frac{\theta^2}{2}\right) - ks^2$$

$$\mathbb{M} = m\begin{array}{c} \begin{array}{cc} s & \theta \end{array} \\ \begin{bmatrix} 1 & l \\ l & l^2 \end{bmatrix} \end{array} \qquad \mathbb{V} = \begin{bmatrix} 2k & 0 \\ 0 & mgl \end{bmatrix}$$

Solve $|M\omega^2 - V| = 0$

$$\begin{vmatrix} m\omega^2 - 2k & ml\omega^2 \\ ml\omega^2 & ml^2\omega^2 - mgl \end{vmatrix} = 0$$

$$(m\omega^2 - 2k)(ml^2\omega^2 - mgl) - m^2l^2\omega^4 = 0$$

$$m^2 l^2 \omega^4 - (2kml^2 + m^2 g l)\omega^2 + 2kmgl - m^2 l^2 \omega^4 = 0$$

$$(2kl + mg)\,\omega^2 = 2kg$$

$$\boxed{\omega^2 = \frac{2kg}{2kl + mg}}$$

$k \to 0$, $\omega^2 \to 0$ ✓

$k \to \infty$ $\omega^2 \to g/l$ ✓

$l \to 0$ $\omega^2 \to 2k/m$ ✓

$m, l \to \infty$ $\omega^2 \to 0$ ✓

3. Say m_1 for $j = 2n+1$, m_2 for $j = 2n$

Eqn of motion from Newton

$$m_1: \quad m_1 \ddot{z}_{2n+1} = k(z_{2n+2} - z_{2n+1}) - k(z_{2n+1} - z_{2n})$$

$$= k(z_{2n+2} + z_{2n} - 2z_{2n+1})$$

$$m_2: \quad m_2 \ddot{z}_{2n} = k(z_{2n+1} - z_{2n}) - k(z_{2n} - z_{2n-1})$$

$$= k(z_{2n+1} + z_{2n-1} - 2z_{2n})$$

Use $z_{2n+1} \quad A e^{ika(2n+1)} \cos(\omega t + \phi)$

$$z_{2n} = B e^{ika(2n)} \cos(\omega t + \phi)$$

So

$$-\omega^2 m_1 A e^{ika(2n+1)} = k(B e^{ika(2n+2)} + B e^{ika(2n)} - 2A e^{ika(2n+1)})$$

$$-\omega^2 m_1 A = k(B e^{ika} + B e^{-ika} - 2A)$$

$$\boxed{-\omega^2 m_1 A = 2k(B \cos ka - A)}$$

and

$$-\omega^2 m_2 B e^{ika(2n)} = k(A e^{ika(2n+1)} + A e^{ika(2n-1)} - 2B e^{ika(2n)})$$

$$-\omega^2 m_2 B = k(A e^{ika} + A e^{-ika} - 2B)$$

$$\boxed{-\omega^2 m_2 B = 2k(A \cos ka - B)}$$

Take ratio: $X \equiv \frac{A}{B}$

$$\frac{m_1}{m_2} X = \frac{\cos ka - X}{X \cos ka - 1}$$

$$m_1 x\,(x\cos ka - 1) = m_2\,(\cos ka - x)$$

$$m_1 \cos ka\, x^2 - (m_1 - m_2)x - m_2 \cos ka = 0$$

$$x = \frac{1}{2m_1\cos ka}\left[m_1 - m_2 \pm \sqrt{(m_1-m_2)^2 + 4m_1 m_2 \cos^2 ka}\right]$$

$$\omega^2 = -\frac{2k}{m_2}\left(x\cos ka - 1\right)$$

$$= -\frac{2k}{m_2}\left[\frac{1}{2m_1}\left(m_1 - m_2 \pm \sqrt{}\right) - 1\right]$$

$$= -\frac{k}{m_1 m_2}\left[m_1 - m_2 \pm \sqrt{} - 2m_1\right]$$

$$\boxed{\omega^2 = \frac{k}{m_1 m_2}\left[m_1 + m_2 \pm \sqrt{(m_1-m_2)^2 + 4m_1 m_2 \cos^2 ka}\right]}$$

Or

$$\sqrt{} = m_1^2 + m_2^2 - 2m_1 m_2 + 4m_1 m_2 \tfrac{1}{2}(1 + \cos 2ka)$$

$$= m_1^2 + m_2^2 + 2m_1 m_2 \cos 2ka$$

$$\boxed{\omega^2 = k\left[\frac{1}{m_1} + \frac{1}{m_2} \pm \sqrt{\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{2}{m_1 m_2}\cos 2ka}\right]}$$

Check: $m_1 = m_2$

$$\omega^2 = \frac{k}{m^2}\left[2m \pm 2m\cos ka\right] = \frac{2k}{m}(1 \pm \cos ka) \checkmark$$

$m_2 \to 0$: $$\omega^2 \to k\left[\frac{1}{m_1} + \frac{1}{m_2} \pm \frac{1}{m_2}\sqrt{1 + \frac{2m_2}{m_1}\cos 2ka + \frac{m_2^2}{m_1^2}}\right]$$

$$\frac{1}{m_2}\left(1 + \frac{m_2}{m_1}\cos 2ka\right)$$

$$\omega^2 \to \frac{k}{m_1}(1 - \cos 2ka)$$

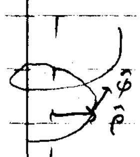
4. a) $T = \frac{1}{2}m\left(\dot{z}^2 + a^2\dot{\phi}^2\right) + \frac{1}{2}m\left(b^2+a^2\right)\dot{\phi}^2$

$V = mgz = mgb\phi$

$$\boxed{L = \frac{1}{2}m\left(a^2+b^2\right)\dot{\phi}^2 - mgb\phi}$$

b) $L = L_0 + m\vec{v}\cdot(\vec{\omega}\times\vec{r}) + \frac{1}{2}m\left|\vec{\omega}\times\vec{r}\right|^2$

$$\vec{r} = a\hat{\rho} + z\hat{z}$$
$$\vec{\omega} = \omega\hat{z}$$



$$\vec{\omega}\times\vec{r} = \omega a\left(\hat{z}\times\hat{\rho}\right) = \omega a\hat{\phi}$$

$$\vec{v} = a\dot{\phi}\hat{\phi} + \dot{z}\hat{z} = \dot{\phi}\left(a\hat{\phi} + b\hat{z}\right)$$

$$\vec{v}\cdot(\vec{\omega}\times\vec{r}) = \omega a^2\dot{\phi}$$

$$\left|\vec{\omega}\times\vec{r}\right|^2 = \omega^2a^2$$

$$\boxed{L = \frac{1}{2}m\left(a^2+b^2\right)\dot{\phi}^2 - mgb\phi + m\omega a^2\dot{\phi}} + \underbrace{\frac{1}{2}m\omega^2a^2}_{\text{Total derivative, can drop}}$$

c) $\frac{\partial L}{\partial\dot{\phi}} = m\left(a^2+b^2\right)\dot{\phi} + m\omega a^2$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = m\left(a^2+b^2\right)\ddot{\phi} + m\dot{\omega}a^2$$

$$\frac{\partial L}{\partial\phi} = -mgb$$

So $m\left(a^2+b^2\right)\ddot{\phi} + m\dot{\omega}a^2 + mgb = 0$

If $\ddot{\phi} = 0$, then $\mathring{\omega} a^2 + gb = 0$

$$\boxed{\mathring{\omega} = -\frac{gb}{a^2}}$$