Lecture 27

Strings / Sturm-Liouville problems

Seen that

1) Modes ρ_n form orthonormal, complete set
2) Variational principle can be used to approximate ρ_n's & ω_n's
3) Green's function describes response to external drive... also contains all information about modes
4) Two ways to get Green's function

Last thing: perturbation theory

Should be familiar from quantum mechanics

Idea: suppose we can solve $(L_0 - \omega^2 \sigma)\rho = 0$

Solutions $\rho_n^0(x)$ ← for some τ, σ, v $\quad L_0 = -\frac{d}{dx}\tau\frac{d}{dx} + v$

But want to solve $(L_0' - \omega^2\sigma)\rho = 0$

$$L_0' = L_0 + \sigma h(x) \equiv L_0 + L_1$$

$h(x)$ small

Then have $(L_0 - \omega^2 \sigma)\rho = -L_1 \rho$

Recall that $(L_0 - \omega^2\sigma)\rho = f(x)$

solved by $\rho(x) = \int G_\omega(x,y) f(y)\, dy$

→ why we introduced G in first place

So, must have here

$$\rho(x) = -\int G(x,y)\, L_1 \rho(y)\, dy$$

Not really a solution, since ρ appears on both sides

But still useful. Substitute $G(x,y) = \sum_{m=1}^{\infty} \frac{\rho_m^0(x)\rho_m^0(y)}{\omega_m^2 - \omega^2}$

$$\rho(x) = \sum \rho_m(x) \frac{1}{\omega^2 - \omega_m^2} \int \rho_m^0(y) L_1 \rho(y)\, dy$$

Form $\rho(x) = \sum_m c_m \rho_m^0(x)$

Since L_1 small, expect $\rho(x)$'s close to $\rho_m^0(x)$'s

Let's find solution close to $\rho_n^0(x)$

Choose normalization so $\langle \rho | \rho_n^0 \rangle = 1$

$$\Rightarrow \quad C_n = 1$$

$$\rho(x) = \rho_n^0(x) + \underbrace{\sum_{m\neq n} \rho_m^0(x) \frac{1}{\omega^2 - \omega_m^2} \int \rho_m^{(0)}(y) L_1 \rho(y)\, dy}_{\text{small, } O(h)}$$

Since correction already small,

approximate $\rho \approx \rho_n^0$ in integral

Then $\rho(x) = \rho_n^{(0)}(x) + \sum_{m\neq n} \rho_m^{(0)} \frac{1}{\omega^2 - \omega_m^2} \int \rho_m^{(0)} \sigma h \rho_n^{(0)}\, dy$

$$\boxed{\rho(x) \cong \rho_n^{(0)}(x) + \sum_{m\neq n} \rho_m^{(0)}(x) \frac{1}{\omega^2 - \omega_m^2} \langle m | h | n \rangle}$$

Gives lowest-order correction to ρ

Can also get correction to frequency ω^2:

We had $\rho = \sum_m \rho_m^0 \frac{1}{\omega^2-\omega_m^2} \int dy\, \rho_m^{(0)}(y) L_1 \rho(y)$

$= \sum_n \rho_m^0 c_m$

We chose $c_n = 1$, so must have

$$\frac{1}{\omega^2-\omega_n^2} \int dy\, \rho_n^0 L_1 \rho = 1$$

$$\omega^2 = \omega_n^2 + \underbrace{\int dy\, \rho_n^0 L_1 \rho}_{\substack{\text{already small,}\\ \text{so approximate}\\ \rho \approx \rho_n^{(0)}}}$$

$$\omega^2 = \omega_n^2 + \int dy\, \rho_m^{(0)}(y)\, \sigma h\, \rho_n^0(y)$$

$$\boxed{\omega^2 \equiv \omega_n^2 + \langle \rho_n^0 | h | \rho_n^0 \rangle}$$

Compare to quantum mechanics:

If $H = H_0 + h$

then $\psi_n \approx \psi_n^0 + \sum_{m \neq n} \frac{\langle \psi_m^0 | h | \psi_n^0 \rangle}{E_n^0 - E_m^0}$

$$E_n \approx E_n^0 + \langle \psi_n^0 | h | \psi_n^0 \rangle$$

Same for $\omega^2 \leftrightarrow E$

Won't bother doing anything with this.
Point is: perturbation theory isn't particular to QM
applies to any Sturm-Liouville system

(Could do perturbation theory in E&M, if you wanted.)

So, that's it for course material!

Q: Was Ch 7 stuff useful?

Look at syllabus again, to review:

Ch 1 Newton's Laws
- Good for unconstrained systems
- Orbits
- Scattering

Important * Ch 3: Lagrangians
- Good for dealing with constraints
- Hamilton's principle
- Calculus of variations
- Lagrange multipliers

Ch 4: Small oscillations
- Simple systems: Taylor expand eqn of motion around equilibrium, get harmonic oscillator
- Bigger systems: Use matrix techniques, get normal modes

* Using Coriolis force in Earth frame

Ch 2: Rotating frames
- Use when frame rotation specified

Ch 5: Rigid bodies
- Use when you need to solve for rotation

* Inertia tensor

Euler eqns

Euler angles

General application: understanding how to transform between frames is important

(often see Euler angles in QM)

Ch 6: Hamiltonian mechanics
- Useful when you have conserved momenta
- Hamilton-Jacobi theory: can solve any seperable system
- Strong connections to QM

Ch 7: String
- Simple example of Sturm-Liouville eqn
  Common in all physics
- Modes, variational theory, Green's functions

Same techniques apply to many situations

$\vec{F} = m\vec{a}$

$\vec{p} = m\vec{v}$

$T = \frac{1}{2} m v^2$

$E = T + V$

$R_{cm} = \frac{1}{M} \sum_i m_i \vec{r}_i$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

$L = T - V$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$

$\delta S = \delta \int L\,dt = 0$

$T = \frac{1}{2} \sum_{ij} m_{ij} \dot{\eta}_i \dot{\eta}_j$

$V = \frac{1}{2} \sum_{ij} v_{ij} \eta_i \eta_j$

$\mathbb{V}\vec{\rho} = \omega^2 \mathbb{M} \vec{\rho}$

$\left(\frac{d}{dt}\right)_{inert} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega} \times$

$F_{cent} = m \omega^2 r_\perp \hat{r}_\perp$

$I_{ij} = \int \rho(\vec{r}) [r^2 \delta_{ij} - r_i r_j] d^3 r$

$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

$L_i = \sum_j I_{ij} \omega_j$

$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$

$H = T + V = E$ for conservative system

$\dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}, \quad \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$

$-\frac{d}{dx} \tau \frac{d\rho}{dx} + v\rho = \omega^2 \sigma \rho$

$\left[-\frac{d}{dx} \tau \frac{d}{dx} + v - \omega^2 \sigma\right] G(x,y) = \delta(x-y)$