Lecture 26

More on Green's function

In general, Green's function = response to "point source"

For string, solution to inhomogeneous Sturm-Liouville problem

$$(L_0 - \omega^2 \sigma)\, G_\omega(x,y) = \delta(x-y)$$

$$L_0 = -\frac{d}{dx}\tau(x)\frac{d}{dx}$$

ω = oscillation freq of source

Last time, saw one way to express solution:

$$G_\omega(x,y) = \sum_{n=1}^{\infty} \frac{\rho_n(x)\rho_n(y)}{\omega_n^2 - \omega^2}$$

$\rho_n(x)$ = solution to homogenous problem

$$(L_0 - \omega_n^2 \sigma)\rho_n = 0$$

Clever way to remember:

Formally, write

$$G_\omega = \frac{1}{L_0(x) - \omega^2\sigma}\,\delta(x-y)$$

Use $\delta(x-y) = \sum_n \rho_n(x)\rho_n(y)\sigma(x)$ (completeness)

$$G_\omega = \frac{1}{L_0 - \omega^2\sigma}\sum_n \rho_n(x)\rho_n(y)\sigma(x)$$

Know $L_0\rho_n = \omega_n^2\sigma\rho_n$: $L_0 \to \omega_n^2\sigma$

So $G_\omega \to \sum_n \frac{\rho_n(x)\rho_n(y)\sigma(x)}{(\omega_n^2-\omega^2)\sigma(x)} = \sum_n \frac{\rho_n(x)\rho_n(y)}{\omega_n^2-\omega^2}$

see G written $\frac{1}{L_0 - \omega^2 \sigma}$ "1" = identity operator $\delta(x-y)$

Today, see another technique to get G
- specific to 1D problems but most PDE's reduce to 1D problem via separation

Suppose we know general solution to S-L eqn:

$$(L_0 - \omega^2 \sigma) u = 0$$

for any ω^2... don't worry about boundary conditions

Example: uniform string has general solution
$$u(x) = \alpha \sin(kx + \phi)$$
$$k = \frac{\omega}{c}$$

For any ω, can construct solution $u_1(x)$ that satisfies BC at $x = a$
(recall $a < x < b$)

Uniform string $[0, l]$: $u_1(x) = \alpha \sin kx$

Note that for $x < y$, Green's fcn also satisfies
$$(L_0 - \omega^2 \sigma) G(x,y) = 0$$

and satisfies BC on $x < a$

By uniqueness must have $G_\omega(x,y) = A u_1(x)$

for $x<y$, write $G_\omega^<(x,y) = A u_1(x)$

Can also find solution $u_2(x)$ that satisfies BC at $x=b$ (but not a, unless $\omega=\omega_n$)

Same reasoning, have $G_\omega^>(x,y) = B u_2(x)$ for $x>y$

So if we figure out A+B, would have solution for all x

- Note A+B generally depend on y

To relate, look at $(L_0 - \omega^2\sigma)G = \delta(x-y)$

$$-\frac{d}{dx}\tau\frac{dG}{dx} + vG - \omega^2\sigma G = \delta(x-y)$$

Integrate from $x=y-\varepsilon$ to $x=y+\varepsilon$, ε small

$$-\int_{y-\varepsilon}^{y+\varepsilon}\frac{d}{dx}\tau\frac{dG}{dx}\,dx + \int_{y-\varepsilon}^{y+\varepsilon}[v-\omega^2\sigma]G\,dx = \int_{y-\varepsilon}^{y+\varepsilon}\delta(x-y)\,dx$$

Recall $G(x,y)$ = response at x to point force at y

Under stable conditions, can't get ∞ response to finite force,

So $G(x,y)<\infty$, and 2^{nd} term $\to 0$ as $\varepsilon\to 0$

Could have $\frac{dG}{dx}$ discontinuous so 1^{st} term remains

Leaves $\displaystyle -\left[\tau(x)\frac{dG(x,y)}{dx}\right]\Bigg|_{x=y-\varepsilon}^{x=y+\varepsilon} = 1$

Assume τ is continuous at $x=y$,

$$= -\tau(y)\left[\frac{dG^>(x,y)}{dx}\bigg|_{x=y} - \frac{dG^<(x,y)}{dx}\bigg|_{x=y}\right] = 1$$

Gives boundary condition at $x=y$
along with $G^<(x,y) = G^>(x,y)$ continuity

Plug in $G^<(x,y) = A u_1(x)$

$$G^>(x,y) = B u_2(x)$$

$$\begin{cases} A u_1(y) = B u_2(y) \\ B u_2'(y) - A u_1'(y) = -\dfrac{1}{\tau(y)} \end{cases}$$

Solve:

$$A = B\frac{u_2}{u_1}$$

$$B\left(u_2' - \frac{u_2}{u_1}u_1'\right) = -\frac{1}{\tau}$$

$$B(u_1 u_2' - u_2 u_1') = -\frac{u_1}{\tau}$$

$$B = -\frac{u_1(y)}{\tau W}$$

$W =$ Wronskian
$= u_1(y)u_2'(y) - u_2(y)u_1'(y)$

and $\displaystyle A = -\frac{u_2(y)}{\tau W}$

In fact, have $\tau(y)\,W(y) = \text{const}$:

Know $(L_0 - \omega^2\sigma)u_1 = (L_0 - \omega^2\sigma)u_2 = 0$

$$u_2(L_0 - \omega^2\sigma)u_1 - u_1(L_0-\omega^2\sigma)u_2 = 0$$

$$u_2 L_0 u_1 - u_1 L_0 u_2 = 0$$

$$-u_2 \frac{d}{dx}\tau u_1' + u_1 \frac{d}{dx}\tau u_2' = 0$$

$$\frac{d}{dx}\left[-\tau u_2 u_1' + \tau u_1 u_2'\right] = 0$$

$$\frac{d}{dx}\left[\tau W(x)\right] = 0$$

So $\tau W(x) = C$ constant

Therefore have simple expression for G:

$$G_\omega(x,y) = \begin{cases} A u_1(x) = -\dfrac{u_1(x)\,u_2(y)}{C} & x < y \\ B u_2(x) = -\dfrac{u_2(x)\,u_1(y)}{C} & x > y \end{cases}$$

Write compactly as

$$\boxed{G_\omega(x,y) = -\frac{u_1(x_<)\,u_2(x_>)}{C}}$$

$x_<$ = smaller of x, y

$x_>$ = larger of x, y

Example: uniform string $\sigma = \tau = \text{const}$

$$u_1(x) = \alpha \sin kx \qquad k = \frac{\omega}{c}$$

$$c = \sqrt{\frac{\tau}{\sigma}}$$

$$u_2(x) = \beta \sin k(x-\ell)$$

$$W = u_1' u_2 - u_2' u_1$$

$$= k\alpha \cos kx \cdot \beta \sin k(x-\ell) - k\beta \cos k(x-\ell) \cdot \alpha \sin kx$$

$$= k\alpha\beta \Big[\cos kx \left(\sin kx \cos k\ell - \cos kx \sin k\ell \right) - \sin kx \left(\cos kx \cos k\ell + \sin kx \sin k\ell \right) \Big]$$

$$= k\alpha\beta \Big[\cos kx \sin kx \cos k\ell - \cos^2 kx \sin k\ell - \sin kx \cos kx \cos k\ell - \sin^2 kx \sin k\ell \Big]$$

$$= -k\alpha\beta \sin k\ell$$

So

$$G(x,y) = -\frac{\alpha \sin(kx_<) \cdot \beta \sin k(\ell - x_>)}{\tau(-k\alpha\beta \sin k\ell)}$$

$$\boxed{G_\omega = \frac{\sin kx_< \sin k(\ell - x_>)}{k\tau \sin k\ell}} \qquad k = \frac{\omega}{c}$$

Compare to previous form:

$$G_\omega = \frac{2}{\ell\sigma} \sum_{n=1}^{\infty} \frac{\sin k_n x \sin k_n y}{\omega_n^2 - \omega^2}$$

$$k_n = \frac{\omega_n}{c} = \frac{n\pi}{\ell}$$

Looks rather different!

Verify they agree:

New expression has poles at $k = \frac{n\pi}{\ell}$

$$\Rightarrow \omega = \frac{n\pi c}{\ell} = \omega_n \checkmark$$

Residue at $\omega = \omega_n$:

$$\lim_{\omega \to \omega_n} (\omega - \omega_n) G_\omega(x, y)$$

$$c\left(k - \frac{n\pi}{\ell}\right) \frac{-\sin k x_< \sin k(\ell - x_>)}{k\tau \sin k\ell}$$

Take $k = \frac{n\pi + \varepsilon}{\ell}$

$$= c \frac{\varepsilon}{\ell} \frac{\sin k_n x_< \sin k_n(\ell - x_>)}{k_n\tau \sin(n\pi + \varepsilon)}$$

Know, $\sin(n\pi + \varepsilon) \to (-1)^n \varepsilon$

Also $\sin k_n(\ell - x_>) = \sin k_n\ell \cos k_n x_> - \cos k_n\ell \sin k_n x_>$

$$= -\cos n\pi \sin k_n x_>$$

$$= -(-1)^n \sin k_n x_>$$

So residue $= -\frac{c\varepsilon}{\ell} \cdot \frac{\sin k_n x_< (-1)^n \sin k_n x_>}{k_n \tau (-1)^n \varepsilon}$

$$= -\frac{c}{\ell k_n \tau} \sin k_n x_< \sin k_n x_>$$

While in previous form,

$$\lim_{\omega \to \omega_n} (\omega - \omega_n) G_\omega = \frac{2}{l\sigma} \frac{\sin k_n x \sin k_n y}{\omega_n + \omega}$$

$$= -\frac{1}{l\sigma\omega_n} \sin k_n x \sin k_n y$$

$$= -\frac{1}{l\sigma k_n c} \sin k_n x \sin k_n y$$

$$\sigma = \frac{\tau}{c^2}$$

$$= -\frac{c}{l\tau k_n} \sin k_n x \sin k_n y \quad \text{same} \checkmark$$

Theorem from complex analysis:

If two functions agree at ∞ set of points that tend to ∞, then functions are identical.

Quite powerful technique.

Try to relate to what you see in E&M:

Green's function in free space $G(\vec{x}, \vec{x}')$

= potential at $\vec{x}$ due to point charge at $\vec{x}'$

= Solution to $\nabla_x^2 G(\vec{x}, \vec{x}) = \delta^3(\vec{x} - \vec{x}')$

Like our situation, but

i) 3D, not 1D

ii) $\omega^2 = 0$: electrostatics

* Just a sketch to illustrate ideas!

Jackson gives $G(\vec{x}, \vec{x}') = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$

r = spherical radius $|\vec{x}|$
θ = polar angle

See this is combination of our two methods:

$\nabla^2\phi$ separates into eqns for r, γ:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial\phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

gives $\phi = \frac{1}{r} u(r) P(\theta)$

with $\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\, P') + \ell(\ell+1) P = 0$

$$u'' - \frac{\ell(\ell+1)}{r^2} u = 0$$

sep const $\ell(\ell+1)$

Solutions $P_\ell(\cos\theta)$ – Legendre polynomials.

$$\frac{1}{r} u_\ell(r) = r^\ell, \; r^{-\ell-1}$$

Green function in r: $\frac{1}{C_\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}}$

Sum over modes for θ:

$$\sum_{\ell=0}^{\infty} \frac{P_\ell(\cos\theta) P_\ell(\cos\theta')}{\ell(\ell+1)} \leftarrow \omega_n^2, \quad \omega^2 = 0$$

But we define coords so $\theta' = 0$

$P_\ell(\cos\theta') = 1$

Combine to $$G(\vec{x},\vec{x}') = \sum_{\ell} C_\ell \frac{r_<^\ell}{r_>^{\ell+1}} \frac{P_\ell(\cos\theta)}{\ell(\ell+1)}$$

$$= \sum_{\ell} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

All the same ideas!