Lecture 25

Sturm-Liouville eqn: $-\frac{d}{dx}(\tau\rho') + v\rho = \omega^2\sigma\rho$

Last time saw that solutions $\rho_n(x)$ formed complete, orthonormal basis for functions on string

$\Rightarrow$ any excitation of string can be expressed as sum of modes

Used fact that functional

$$\omega^2[\rho] = \frac{\int \tau\rho'^2 + v\rho^2\,dx}{\int \sigma\rho^2\,dx}$$ is stationary for $\rho = \rho_n$

Can express this in a different way:

Use $\rho = \sum a_n\rho_n$

$$\rho^2 = \sum_n\sum_m a_n a_m \rho_n\rho_m$$

So $$\int \sigma\rho^2 dx = \sum_{nm} a_n a_m \underbrace{\int \rho_n\rho_m\sigma\,dx}_{\delta_{nm}}$$

$$= \sum_{nm} a_n^2$$

Also $\rho'^2 = \sum_{nm} a_n a_m \rho_n'\rho_m'$

and $$\int \tau\rho_n'\rho_m'\,dx = \underbrace{\rho_m\tau\rho_n'\Big|_a^b}_{=0} - \int_a^b \rho_m \frac{d}{dx}(\tau\rho_n')\,dx$$

S...

So numerator →

$$\sum_{nm} a_n a_m \int_a^b \rho_m \left[-\frac{d}{dx}(\tau \rho_n') + v\rho_n \right] dx$$

$$\int_a^b \rho_m (\omega_n^2 \tau \rho_n) dx$$

$$\omega_n^2 \int \rho_m \rho_n \tau dx$$

$$\omega_n^2 \delta_{nm}$$

$$= \sum_n a_n^2 \omega_n^2$$

So we have $\boxed{\omega^2[\rho] = \dfrac{\sum_n a_n^2 \omega_n^2}{\sum_n a_n^2}}$

weighted average of different modes
Like expectation value in QM

In QM, this is basis for variational method:
Pick ρ to make $\omega^2[\rho]$ as small as possible
then $\omega^2[\rho] \approx \omega_1^2$ smallest frequency

See this works for any S-L problem

Furthermore, convergence is rather good:

Say $\rho = a_1 \rho_1 + \sum_{n=2}^{\infty} \varepsilon_n \rho_n$

$$\varepsilon_n \ll 1$$

$$a_1^2 = 1 - \sum \varepsilon_n^2 \approx 1,$$

Then $$\omega^2[\rho] = \frac{a_1^2\omega_1^2 + \sum_{n=2}^{\infty} \varepsilon_n^2 \omega_n^2}{a_1^2 + \sum_{n=2} \varepsilon_n^2}$$

$$= \frac{\omega_1^2 + \sum \varepsilon_n'^2 \omega_n^2}{1 + \sum \varepsilon_n'^2} \qquad \varepsilon_n' = \frac{\varepsilon_n}{a_n}$$

$$\approx \left(\omega_1^2 + \sum \varepsilon_n'^2 \omega_n^2\right)\left(1 - \sum \varepsilon_n'^2\right)$$

$$= \omega_1^2 - \sum \varepsilon_n'^2 \omega_1^2 + \sum \varepsilon_n'^2 \omega_n^2 + O(\varepsilon^4)$$

$$= \omega_1^2 + \sum \varepsilon_n'^2 \left(\omega_n^2 - \omega_1^2\right)$$

$$= \omega_1^2 + O(\varepsilon^2)$$

$\Rightarrow$ approximation for ω_1^2 is accurate to order ε^2
So even a relatively inaccurate ρ
can give decent ω_1^2

Get to work through example in HW
- assume you've seen idea already in QM

Turn attention now to Green's function

Have said what we can about modes $\rho_n(x)$
$\rightarrow$ free motion of string

Think now about response to driving force

Consider periodic driving force / unit length

$$F(x,t) = \sigma(x) f(x) \cos\omega t$$

$F(x) = \frac{\text{mass}}{\text{length}} \times \text{acceleration} = \sigma \frac{\partial^2 u}{\partial t^2}$

So string equation becomes

$$\sigma \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\tau \frac{\partial u}{\partial x}\right) - vu + \sigma f \cos\omega t$$

As before, look for steady state solution $u(x,t) = u(x)\cos\omega t$
(book's notation)

$$-\omega^2 \sigma u(x) = \frac{d}{dx}\left(\tau \frac{du}{dx}\right) - vu + \sigma f$$

or $$-\frac{d}{dx}(\tau u') + vu - \omega^2 \sigma u = \sigma f$$

Remember, here ω = arb freq
generally $\neq \omega_n$

Abreviate $L_0 = -\frac{d}{dx}\,\tau\frac{d}{dx} + v$ operator

So eqn is $\boxed{L_0 u - \omega^2 \sigma u = \sigma f}$

Two methods for solving:

I. Eigen-function expansion

$\rho_n(x) =$ solutions to S-L eqn $(f=0)$

Expand $u(x) = \sum c_n \rho_n(x)$

Since $L_0 \rho_n = \omega_n^2 \sigma \rho_n$, eqn becomes

$$\sum_n c_n(\omega_n^2 - \omega^2)\sigma \rho_n = \sigma f$$

Solve for c_n in usual way:

$$\sum_n c_n(\omega_n^2-\omega^2)\underbrace{\int \rho_m \sigma \rho_n\, dx}_{\delta_{mn}} = \underbrace{\int \rho_m \sigma f\, dx}_{\langle \rho_m | f\rangle}$$

So $\quad c_m = \dfrac{\langle \rho_m | f\rangle}{\omega_m^2-\omega^2}$

$$\boxed{u(x) = \sum_{n=1}^{\infty} \rho_n(x)\, \frac{\langle \rho_n | f\rangle}{\omega_n^2-\omega^2}}$$

Easy enough

Notes:

- Diverges when $\omega = \omega_n$ (unless $\langle \rho_n | f\rangle = 0$)
  No damping, so infinite response on resonance
- Requires evaluation of $\langle \rho_n | f\rangle$ for all n
  Often inconvenient

Green's fcn eases this difficulty:

Can always write $f(x) = \int_a^b dy\, f(y)\,\delta(x-y)$

= sum of point forces

If we knew response $u_y'(x)$ to point force at y,

Then by linearity,

$$u(x) = \int dy\, \sigma(y) f(y)\, u_y(x)$$

Usually write $u_y(x) \to G(x,y) \equiv$ Green's function

Solution to $(L_0 - \omega^2 \sigma) G(x,y) = \delta(x-y)$

Remind ourselves it depends on ω: $G_\omega(x,y)$

See that if we had $G_\omega(x,y)$, could get response to <u>any</u> f

$$u(x) = \int dy\, f(y)\, G_\omega(x,y)\, \sigma(y)$$

Can get G from solution we already have:

$$u(x) = \sum \rho_n(x) \frac{\langle \rho_n | f \rangle}{\omega^2 - \omega_n^2}$$

$$= \sum \frac{1}{\omega^2 - \omega_n^2} \rho_n(x) \int dy\, \rho_n(y) f(y) \sigma(y)$$

$$= \int dy\, f(y) \left[\sum \frac{1}{\omega^2 - \omega_n^2} \rho_n(x) \rho_n(y) \right] \sigma(y)$$

Identify
$$\boxed{G_\omega(x,y) = \sum_{n=1}^{\infty} \frac{\rho_n(x)\rho_n(y)}{\omega^2 - \omega_n^2}}$$

So if we know solutions to free motion of string, can easily construct forced motion

Properties of Green's function

1) Symmetric $G_\omega(x,y) = G_\omega(y,x)$

Symmetry not apparent in physical introduction:
Response at x to point force at y
= response at y to point force at x

2) G_ω depends on boundary conditions (since $\rho_n(x)$'s and ω_n^2's do)

3) As function of ω^2, G_ω can be extended to complex plane.

Has simple poles at eigenvalues ω_n^2
residue $-\rho_n(x)\rho_n(y)$

So if someone handed you closed form function $G_\omega(x,y)$, you could get all ω_n^2's, ρ_n's

$\Rightarrow$ Green's function contains all information about free motion of string

Example: Green's function for simple string:

$T(x) = \text{const}$

$\sigma(x) = \text{const}$

endpoints fixed $x=0$, $x=\ell$

Know solutions from lecture 11/18:

$$\rho_n(x) = \sqrt{\frac{2}{\ell\sigma}}\sin\left(\frac{n\pi x}{\ell}\right) \qquad \omega_n^2 = \left(\frac{n\pi c}{\ell}\right)^2$$

So

$$\boxed{G_\omega(x,y) = \frac{2}{\ell\sigma}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi y}{\ell}\right)}{\left(\frac{n\pi c}{\ell}\right)^2 - \omega^2}}$$

Not much to it. Still too bad it involves ∞ sum

Next time, see another method that can give closed-form solution