Lecture 24 ...

Last time, saw Sturm-Liouville eqn

$$-\frac{d}{dx}\left[\tau(x)\frac{d\rho}{dx}\right] + v(x)\rho = \omega^2 \sigma(x)\rho$$

Common form obtained when seperating PDE's
General equation for strings, tension τ
ext potential $\frac{1}{2}v\rho^2$
mass density σ

Solve on finite intervel $a \le x \le b$
Three restrictions
i) τ, v, σ real
ii) τ, v, σ integrable over intervel
iii) $\tau, \sigma > 0$ on $a < x < b$
Can have $\tau, \sigma \to 0$ on boundary

Allow following boundary conditions
Any combination
i) $\rho = 0$
ii) $\tau \rho' = 0$
iii) periodic

4^{th} also works: $\alpha \rho' = \beta \rho$
But proofs a little harder... see text

Rely on mathematicians for a few things
- BC's satisfied only for discrete values of $\omega^2 \to \omega_n^2$

- As $n \to \infty$, $\omega_n^2 \to \infty$

Physically, clear for string

Mathematically, from fact that if L = S-L operator then $\frac{1}{L-z}$ is compact for some z

From that, show

1) orthogonality of solutions ρ_n
2) completeness of solutions ρ_n

Here ρ_n = solution for $\omega^2 = \omega_n^2$

Orthogonality:

Multiply eqn for ρ_n by ρ_m^*
eqn for ρ_m^* by ρ_n
and subtract:

$$\left[-\rho_m^* \frac{d}{dx}(\tau \rho_n') + v \rho_m^* \rho_n\right]$$
$$- \left[-\rho_n \frac{d}{dx}(\tau \rho_m^{*\prime}) + v \rho_n \rho_m^*\right]$$
$$= (\omega_n^2 - \omega_m^{*2}) \sigma \rho_n \rho_m^*$$

Note

$$\rho_m^* \frac{d}{dx}(\tau \rho_n') = \frac{d}{dx}(\rho_m^* \tau \rho_n') - \tau \rho_n' \rho_m^{*\prime}$$
$$\rho_n \frac{d}{dx}(\tau \rho_m^{*\prime}) = \frac{d}{dx}(\rho_n \tau \rho_m^{*\prime}) - \tau \rho_n' \rho_m^{*\prime}$$

Get

$$\frac{d}{dx}\left[\rho_n \tau \rho_m^{*\prime} - \rho_m^* \tau \rho_n'\right] = (\omega_n^2 - \omega_m^{*2}) \sigma \rho_n \rho_m^*$$

Integrate $\int_a^b dx$:

$$\left[\rho_n \tau \rho_m^{*\prime} - \rho_m^* \tau \rho_n^{\prime}\right]\Big|_a^b = (\omega_n^2 - \omega_m^{*2}) \int_a^b \sigma \rho_n \rho_m^* \, dx$$

For any of our boundary conditions, LHS = 0

$\rho = 0$ ✓

$\tau \rho' = 0$ ✓

periodic ✓

So $\quad (\omega_n^2 - \omega_m^{*2}) \int \sigma \rho_n \rho_m^* \, dx = 0$

If $n = m \qquad \underbrace{\int \sigma |\rho_n|^2 dx}_{\text{positive}}$

set $\quad \omega_n^2 = \omega_n^{*2} \Rightarrow \omega_n^2$ real

* We shall also assume that $\omega_n^2 \geq 0$

$\Rightarrow$ system is stable

OK if $U(x)$ not too large + negative

Order ω_n^2: $\omega_n^2 \leq \omega_{n+1}^2$ always

Can also take ρ_n real

But sometimes useful not to

(e^{ikx} vs $\sin(kx+\phi)$)

If $\omega_n^2 \neq \omega_m^2$, set $\int \rho_m^* \rho_n \sigma(x) \, dx = 0$

orthogonal relative to σ

If $\omega_n^2 = \omega_m^2$ for $m \neq n$, can make orthogonal solutions via Gram-Schmit

Can also normalize ρ as desired

Make $\int_a^b \rho_m^* \rho_n \sigma \, dx = \delta_{mn}$

Can define inner product $\langle \rho_m | \rho_n \rangle = \int_a^b \rho_m^* \rho_n \sigma \, dx$

$$= \delta_{nm}$$

See analog to Ch 4 $\sigma(x) \Leftrightarrow M$

Completeness

Can write $f(x) = \sum a_n \rho_n(x)$

$$a_n = \langle \rho_n | f \rangle$$

Show that $\sum \to f$

Prove this & some other things using variational principle:

Define functional

$$\omega^2[\rho] = \frac{\frac{1}{2}\int_a^b \left[\tau \left(\frac{d\rho}{dx}\right)^2 + v\rho^2\right] dx}{\frac{1}{2}\int_a^b \sigma \rho^2 \, dx} = \frac{I_1}{I_2}$$

For simplicity, take ρ real now

Claim S-L eqn equivalent to $\delta \omega^2 = 0$

Show: $\delta \omega^2 = \frac{\delta I_1}{I_2} - \frac{I_1 \delta I_2}{I_2^2}$

$$= \frac{1}{I_2}\left(\delta I_1 - \frac{I_1}{I_2} \delta I_2\right)$$

$$= \frac{1}{I_2}\left(\delta I_1 - \omega^2 \delta I_2\right)$$

$I_2 > 0$ so $\delta \omega^2 = 0 \Rightarrow \delta I_1 - \omega^2 \delta I_2 = 0$

$$\delta I_1 = \int \tau \rho' \delta\rho' + v\rho\,\delta\rho \, dx$$

$$\delta I_2 = \int \sigma \rho \, \delta\rho \, dx$$

Use $\int_a^b \tau \rho' \, \delta\rho' dx = \left[\delta\rho \, \tau \frac{d\rho}{dx}\right]\Big|_a^b - \int \delta\rho \frac{d}{dx}[\tau\rho'] \, dx$

See $\delta\rho \, \tau\rho' \big|_a^b = 0$ for any of our BC's

So if $\delta\omega^2 = 0$, need

$$\int \delta\rho \left[-\frac{d}{dx}(\tau\rho') + v\rho - \omega^2 \sigma \rho \right] dx = 0$$

Requires $[\;] = 0$, $\Rightarrow$ S-L eqn

So, solutions to SL eqn are functions that make $\omega^2[\rho]$ stationary.

Completeness

Start by approximating $f(x) = \sum_{n=1}^{N} \alpha_n \rho_n(x)$

arb. coeffs α

$$\delta_N = \int_a^b \left[f - \sum_{n=1}^{N} \alpha_n \rho_n \right]^2 \sigma \, dx$$

Pick α_n to minimize δ_N

$$\delta_N = \int f^2 \sigma dx - 2 \sum_{n=1}^{N} \alpha_n \underbrace{\int f \rho_n \sigma dx}_{\langle f|\rho_n\rangle \equiv a_n}$$

$$+ \sum_{n,m} \alpha_n \alpha_m \underbrace{\int \rho_n \rho_m \sigma dx}_{\delta_{nm}}$$

$$\delta_N = \int f^2 \sigma\, dx - 2\sum_{n=1}^{N} \alpha_n a_n + \sum_{n=1}^{N} \alpha_n^2$$

Set $\frac{\partial \delta_N}{\partial \alpha_n} = 0 = 2a_n - 2\alpha_n$

$\Rightarrow \boxed{\alpha_n = a_n}$, best coefficients

Want to show that for $\alpha_n = a_n$, $\delta_N \to 0$ as $N \to \infty$

Define $g_N(x) = \frac{1}{\sqrt{\delta_N}}\left[f(x) - \sum_{1}^{N} a_n \rho_n(x) \right]$

Then $\int g_N^2\, dx = \frac{\delta_N}{\delta_N} = 1$

Also $\int_a^b g_N \rho_m \sigma\, dx$

$$= \frac{1}{\sqrt{\delta_N}}\Big[\underbrace{\int f \rho_m \sigma\, dx}_{a_m} - \sum a_n \underbrace{\int \rho_n \rho_m \sigma\, dx}_{\delta_{nm}}\Big]$$

$$[a_m - a_m] = 0$$

g_N orthogonal to ρ_1 to ρ_N

Consider $\omega^2[g_N]$:

Consider all fcns $\bar{\rho}$ with $\langle \rho_n | \bar{\rho} \rangle = 0$, $n = 1 \dots N$

Know $\omega^2[\bar{\rho}] \geq 0$, so expect some $\bar{\rho}_0$ minimizes $\omega^2[\bar{\rho}]$

Since $\delta\omega^2[\bar{\rho}_0] = 0$, must have $\bar{\rho}_0 = \rho_K$ a solution to S-L eqn

$\omega^2[\bar{\rho}_0] = \omega_K^2$

But $\bar{\rho}_0 \neq \rho_1 \ldots \rho_N$

Since ω_n^2's ordered, must have $\omega_K^2 \geq \omega_N^2$

For any other $\bar{\rho} \neq \bar{\rho}_0 \rightarrow \omega^2(\bar{\rho}) \geq \omega^2(\bar{\rho}_0) = \omega_K^2$
$\geq \omega_N^2$

Since g_N is a possible $\bar{\rho}$,

see that $\boxed{\omega^2[g_N] \geq \omega_N^2}$

(Note that $\delta\omega^2[g_N]$ probably not $= 0$
but $\omega^2[g_N]$ still well defined)

Also have
$$\omega^2[g_N] = \frac{\frac{1}{2}\int(\tau g_N'^2 + V g_N^2)dx}{\frac{1}{2}\int \sigma g_N^2 dx}$$

Evaluate $g_N = \frac{1}{\sqrt{\delta_N}}\left(f - \sum a_n \rho_n\right)$:

$$\omega^2[g_N] = \frac{1}{\delta_N}\Big[\int \tau f'^2 + V f^2 \, dx \qquad (1)$$

$$- 2\sum_{n=1}^{N} a_n \int \tau \rho_n' f' + V \rho_n f \, dx \qquad (2)$$

$$+ \sum_{n,m} a_n a_m \int \tau \rho_n' \rho_m' + V \rho_n \rho_m \, dx\Big] \qquad (3)$$

Rewrite first term as $\frac{1}{\delta_N}\,\omega^2[f]\int f^2 \sigma \, dx$

In 3rd term, use integration by parts:

$$\int \tau \rho_n' \rho_m' dx = \underbrace{\rho_m \tau \rho_n' \Big|_a^b}_{=0} - \int \rho_m \frac{d}{dx}(\tau \rho_n') dx$$

So term is

$$\frac{1}{\delta_N} \sum_{n,m} a_n a_m \int \rho_m \underbrace{\left[-\frac{d}{dx}(\tau \rho_n') + v\rho_n \right]}_{= \omega_n^2 \sigma \rho_n} dx$$

$$\omega_n^2 \underbrace{\int \rho_m \rho_n \sigma dx}_{\delta_{mn}}$$

$$\rightarrow \frac{1}{\delta_N} \sum_{n=1}^{N} a_n^2 \omega_n^2$$

Finally, for 2^{nd} term, use

$$\int \tau \rho_n' f' dx = \underbrace{f \tau \rho_n' \Big|_a^b}_{\substack{=0 \\ \text{if } f \text{ also} \\ \text{satisfies BCs}}} - \int f \frac{d}{dx}(\tau \rho_n') dx$$

$$\text{So (2)} \rightarrow -\frac{2}{\delta_N} \sum a_n \int f \underbrace{\left[-\frac{d}{dx}(\tau \rho_n') + v\rho_n \right]}_{\omega_n^2 \sigma \rho_n} dx$$

$$= -\frac{2}{\delta_N} \sum a_n \omega_n^2 \int f \rho_n \sigma \, dx$$

$$= -\frac{2}{\delta_N} \sum a_n^2 \omega_n^2$$

So $\omega^2[g_N] = \frac{1}{\delta_N}\Big[\omega^2[f]\int f^2\sigma\,dx$

$$- \underbrace{\sum_{n=1}^{N}\omega_n^2 a_n^2}_{\text{positive}}\Big] \geq \omega_{N+1}^2$$

So $\frac{1}{\delta_N}\omega^2[f]\int f^2\sigma\,dx \geq \omega_{N+1}^2$

$$\delta_N \leq \frac{\omega^2[f]\,\langle f|f\rangle}{\omega_{N+1}^2}$$

As long as f is integrable, numerator is finite #, indep of N

But we know from math that $\omega_N^2 \to \infty$ as $N \to \infty$

Therefore $\delta_N \to 0$ as $N \to \infty$

and $\lim_{N\to\infty}\sum_{n=1}^{\infty} a_n\rho_n(x) \to f(x)$

Set of solutions to S-L eqn is always complete.

$$f(x) = \int \sum_{n=1}^{\infty}\rho_n(x)\rho_n(x')\,\sigma(x')\,f(x')\,dx'$$

Means $\boxed{\sum_{n=1}^{\infty}\rho_n(x)\rho_n(x')\,\sigma(x') = \delta(x-x')}$

equivalent to completeness