Lecture 23

Finish action-angle example

Method:

For each coord, calculate $J_\sigma = \oint p_\sigma \, dq_\sigma$

$= \text{constant}$ (for separable)

Invert to get $E(J_\sigma\text{'s})$

Then oscillation freq $\nu_\sigma = \dfrac{\partial E}{\partial J_\sigma}$

Looking at Kepler problem

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2m l^2} - \frac{\gamma m}{r} = E < 0$$

$$p_\phi = l = \text{const}$$

$$p_r = \pm\sqrt{2mE + \frac{2\gamma m^2}{r} - \frac{l^2}{r^2}}$$

$$J_\phi = \oint l \, d\phi = 2\pi l$$

$$J_r = \oint \pm\sqrt{\quad} \, dr$$

Worked out integral: $J_r = 2\pi\left(\sqrt{\frac{2m^3\gamma^2}{-E}} - 2l\right)$

Get $E(J\text{'s})$: $J_r = 2\pi\sqrt{\frac{2m^3\gamma^2}{-E}} - 2J_\phi$

$$J_r + 2J_\phi = 2\pi\sqrt{\quad}$$

$$\frac{-E}{2m^3\gamma^2} = \frac{4\pi^2}{(J_r + 2J_\phi)^2}$$

$$\boxed{E = -\frac{8\pi^2 m^3 \gamma^2}{(J_r + 2J_\phi)^2}}$$

Then frequencies are

$$\nu_\phi = \frac{\partial E}{\partial J_\phi} = \frac{32\pi^2 m^3 \gamma^2}{(J_r + 2J_\phi)^3}$$

Express in terms of E: $\quad J_r + 2J_\phi = 2\pi \left(\frac{2m^3\gamma^2}{-E}\right)^{1/2}$

$$\nu_\phi = 32\pi^2 m^3 \gamma^2 \frac{1}{(2\pi)^3} \left(\frac{-E}{2m^3\gamma^2}\right)^{3/2}$$

$$= \frac{4}{\pi} \frac{1}{\sqrt{m^3\gamma^2}} \left(\frac{-E}{2}\right)^{3/2}$$

$$\boxed{\nu_\phi = \frac{\sqrt{2}}{\pi} \frac{1}{\gamma m^{3/2}} (-E)^{3/2}}$$

Agrees w/ result from Ch 1:

$$\tau = 2\pi a^{3/2} \gamma^{-1/2} \qquad a = \frac{\gamma m}{-2E}$$

$$= 2\pi \left(\frac{\gamma m}{-2E}\right)^{3/2} \frac{1}{\sqrt{\gamma}} = \frac{\pi}{\sqrt{2}} \gamma m^{3/2} (-E)^{-3/2} \quad \checkmark$$

Note: Bohr's original quantum theory said to quantize action $J_\sigma = n_\sigma h$

Planck's constant

Here do get correct energy levels for hydrogen:

$$\gamma = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{m}$$

$J_r, J_\phi \rightarrow$ integers

$$E = -\frac{E_0}{(n_r + 2n_\phi)^2} = -\frac{E_0}{n^2} \qquad n = \text{integer}$$

But degeneracy & relation between E & l isn't right

In H atom, can't fix $\vec{L} \parallel \hat{z}$

Different states have $\vec{L}$'s in different directions

Remedy in HW 6.16

That finishes Ch 6, and mechanics of particles in general

Last lectures: mechanics of continuous media

Not on qualifier, but methods similar to quantum, E&M (likely on final)

Here simplest possible system: 1D string

Easiest math

Simplest interpretation

Today: wave eqn on string

Recall theory for continuous medium:

Lagrangian $L = \int \mathcal{L}\, dx$

$$\mathcal{L} = \frac{1}{2}\sigma(x)\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial u}{\partial x}\right)^2$$

σ = mass / length

τ = tension

u = string displacement

$$\delta L = 0 \Rightarrow \quad \sigma \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[\tau \frac{\partial u}{\partial x}\right] = 0$$

Simplest case: $\tau = \text{const}$, $\sigma = \text{const}$

$$\frac{\tau}{\sigma} = c^2$$

Gives wave equation $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

Look for solutions

$$u(x,t) = C\rho(x)\cos(\omega t+\phi) = \rho(A\cos\omega t + B\sin\omega t)$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$$

So

$$-\frac{\omega^2}{c^2}\rho = \frac{d^2\rho}{dx^2}$$

$$k = \frac{\omega}{c}$$

$$\rho'' + k^2\rho = 0 \qquad \text{Harmonic oscillator}$$

Solutions $\rho(x) = A\sin(kx+\theta)$

Boundary conditions restrict value of k, θ

For instance, if $u(0) = u(l) = 0$

Need $\theta = 0$

$$k = \frac{n\pi}{l} \qquad n = 1, 2, \ldots$$

and therefore $\omega = \omega_n = \frac{n\pi c}{l}$

Can pick A however we like:

Impose normalization condition

$$\int_0^l \rho_n(x)\rho_m(x)\,\sigma\,dx = \delta_{nm} \Rightarrow A = \sqrt{\frac{2}{l\sigma}}$$

Recall $\vec{\rho}_n^{\,+} M \vec{\rho}_m = \delta_{nm}$ from Ch 4

General solution will be superposition of modes

$$u(x,t) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\ell\sigma}} \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \cos\frac{n\pi c t}{\ell} + b_n \sin\frac{n\pi c t}{\ell}\right]$$

Get a_n & b_n from initial conditions:

Say $u(x,0) = f(x)$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Then

$$f(x) = \sum_n \sqrt{\frac{2}{\ell\sigma}}\, a_n \sin\frac{n\pi x}{\ell} = \sum_n a_n \rho_n(x)$$

$$\int \rho_m(x) f(x)\,\sigma\,dx = \sum_n a_n \underbrace{\int \rho_m(x)\rho_n(x)\,\sigma\,dx}_{=\delta_{mn}},$$

So $a_m = \int \rho_m(x) f(x)\,\sigma\,dx$

Also

$$g(x) = \sum_n \rho_n(x)\,\omega_n b_n$$

which gives $b_m = \frac{1}{\omega_m}\int \rho_m(x) g(x)\,\sigma\,dx$

Explicit general solution

But does raise some questions:

- Does infinite series for u converge?
  Do 2nd derivatives converge?
- Do series for f & g converge to actual f & g?
- Can we apply this method to more general string eqn $\sigma = \sigma(x)$, $T = T(x)$?

Will address these questions.

One aside:

May have seen that solution to 1D wave eqn can also be expressed

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

arb functions ϕ, ψ

Rather different from what we found

Text §39 reconciles ... interesting

But I want to talk about how method generalizes

General string eqn:

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial u}{\partial x}\right] = \sigma \frac{\partial^2 u}{\partial t^2}$$

Generalize a bit more:

Say that string moves in external potential

$$V[u] = \int \tfrac{1}{2} v(x) u^2 \, dx$$

Extra restoring force/unit length $= -v(x)u(x)$

Imagine springs pulling string toward equilibrium, in addition to tension

Not very likely, but math is useful for other situations

Eqn of motion $\rightarrow$ $\frac{\partial}{\partial x}\left[\tau \frac{\partial u}{\partial x}\right] - v(x)u = \sigma \frac{\partial^2 u}{\partial t^2}$

Again look for solution $u(x,t) = \rho(x)\cos(\omega t + \phi)$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$$

So $$\boxed{-\frac{d}{dx}\left[\tau \frac{d\rho}{dx}\right] + v\rho = \omega^2 \sigma \rho}$$

Sturm-Liouville equation

Very general, since τ, v, σ = arb fcns of x

Example:

a) τ, σ = const, $v = 0$: Harmonic oscillator

b) $\tau = \sigma = x \qquad v = \frac{m^2}{x} \qquad m = \text{const}$

$$-\frac{d}{dx}\left[x\rho'\right] + \frac{m^2}{x}\rho = \omega^2 x \rho$$

$$-x\rho'' - \rho' + \frac{m^2}{x}\rho = \omega^2 x \rho$$

$$x^2\rho'' + x\rho' + (\omega^2 x^2 - m^2)\rho = 0$$

Bessel's equation solutions $J_m(\omega x)$, $N_m(\omega x)$

c) $\sigma = 1 \qquad \tau = 1 - x^2 \qquad v = \frac{m^2}{1-x^2}$

$$-\frac{d}{dx}\left[(1-x^2)\frac{d\rho}{dx}\right] + \frac{m^2}{1-x^2}\rho = \omega^2 \rho$$

$$(1-x^2)\rho'' - 2x\rho' + \left(\omega^2 - \frac{m^2}{1-x^2}\right)\rho = 0$$

Legendre's equation $\omega^2 = l(l+1)$

solutions $P_l^m(x)$

Most of the special functions you encounter when solving PDE's are solutions of a Sturm-Liouville eqn.

Can prove several useful things about solutions, in general

Say solving on interval $a \le x \le b$

Adopt three restrictions:

i) v, σ, τ real on $a \le x \le b$

ii) $\tau, \sigma > 0$ on $a < x < b$

iii) τ, σ integrable over interval

Can have $\tau, \sigma \to 0$ on endpoints

Also need to restrict boundary conditions, but not too much

Any combination of

i) $\rho = 0$

ii) $\tau \rho' = 0$

iii) $\alpha \rho' = \beta \rho$ (α, β real)

or periodic iv) $\rho(a) = \rho(b)$; $\rho'(a) = \rho'(b)$

Covers almost all situations

In all cases, boundary conditions restrict solutions to Sturm-Liouville eqn to discrete values of ω^2:

$$-\frac{d}{dx}\left[\tau \frac{d\rho_n}{dx}\right] + v\rho_n = \omega_n^2 \sigma \rho_n$$

$n = 1, 2, 3, \ldots$

Need to find allowed ω_n^2 as part of solution

Need to rely on mathematicians for a few things:

$\Rightarrow$ Allowed ω_n^2 are discrete, and

$\omega_n^2 \to \infty$ as $n \to \infty$ (will need this later)

- Physically, this is clear for a string.
- Mathematically, comes from fact that $\frac{1}{L-z}$ is a compact operator for any z (L = differential operator of S-L eqn)

First, show that ρ_n's are orthogonal!

a) $$\frac{d}{dx}\tau\rho_p' - v\rho_p = -\omega_p^2\sigma\rho_p$$

b) $$\frac{d}{dx}\tau\rho_q' - v\rho_q = -\omega_q^2\sigma\rho_q$$

Multiply (a) by ρ_q^*, (b)* by ρ_p, and subtract:

$$\rho_q^*\frac{d}{dx}(\tau\rho_p') - \rho_p\frac{d}{dx}(\tau\rho_q'^*) = \left[(\omega_q^2)^* - (\omega_p^2)\right]\rho_q^*\sigma\rho_p$$

$$+\ \tau\rho_p'\rho_q'^* - \tau\rho_p'\rho_q'^*$$

$$\frac{d}{dx}\left[\rho_q^*\tau\rho_p' - \rho_p\tau\rho_q'^*\right] =$$

Integrate

$$\rho_q^*\tau\rho_p' - \rho_p\tau\rho_q'^*\Big|_a^b = \left[(\omega_q^2)^* - (\omega_p^2)\right]\int_a^b \rho_q^*\sigma\rho_p\,dx$$

Vanishes for any of our boundary conditions

$\rho(a) = 0$ ✓

$\tau(a)\rho'(a) = 0$ ✓

periodic ✓

For general linear $\alpha\rho' = \beta\rho$:

$$\rho_q^*(a)\,\tau(a)\,\rho_p'(a) - \rho_p(a)\,\tau(a)\,\rho_q'^*(a)$$

$$= \tau\left[\rho_q^*(a)\,\frac{\beta}{\alpha}\,\rho_p(a) - \rho_p(a)\,\frac{\beta}{\alpha}\,\rho_q^*(a)\right]$$

$$= 0$$

So $\left[(\omega_q^2)^* - \omega_p^2\right] \int_a^b \rho_q^* \rho_p\, \tau\, dx = 0$

If we take $p = q$, know $\int_a^b |\rho_p|^2 \tau\, dx > 0$

So must have $(\omega_p^2)^* = \omega_p^2 \Rightarrow \omega_p^2$ real

We shall also assume that $\omega_p^2 > 0$

$\Rightarrow$ demand physical system is stable

If $\omega_p^2 \neq \omega_q^2$, see $\int_a^b \rho_q^* \tau \rho_p\, dx = 0$

ρ_p and ρ_q are orthogonal,
relative to $\tau(x)$

Can choose normalization so $\int_a^b \rho_q^* \tau \rho_p\, dx = \delta_{pq}$

(If $\omega_p^2 = \omega_q^2$ for $p \neq q$, orthogonalize
via Gram-Schmit procedure)