Lecture 22

Think I can say what makes H-J method powerful:

H-J eqn is 1^{st} order

$$H\left(\frac{\partial S}{\partial q}, q\right) + \frac{\partial S}{\partial t} = 0$$

So *if* equation separates

$$S = S_1(q_1) + S_2(q_2) + \dots \; S_0(t)$$

then each S_i satisfies first order ODE

Can solve any 1^{st} order ODE by integration

Not true for 2^{nd} & higher order ODEs, such as from Lagrangian

So using H-J method, any separable system can be solved

Coord is separable if q_σ & p_σ appear only as $f(p_\sigma, q_\sigma)$ in $H(p,q)$

Example: Spherical coords (r, θ, ϕ)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r,\theta,\phi)$$

Say $$V = \frac{a(\phi)}{r^2\sin^2\theta} + b(r,\theta)$$

then $$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + b(r,\theta)\right) + \frac{1}{r^2\sin^2\theta}\underbrace{\left(p_\phi^2 + a(\phi)\right)}_{f(\phi, p_\phi)}$$

ϕ separates

Go through:

$$S(r,\theta,\phi,t) = W(r,\theta) + \Phi(\phi) - Et$$

So $$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\Phi'\right)^2\right] + \frac{a(\phi)}{r^2\sin^2\theta} + b(r,\theta) - E = 0$$

$$\frac{r^2\sin^2\theta}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + 2mb - 2mE\right] = -\left[\left(\Phi'\right)^2 + a(\phi)\right]$$

$$= -\alpha_\phi \quad \text{const}$$

So $$\left(\frac{d\Phi}{d\phi}\right)^2 + a = \alpha_\phi$$

$$\Phi = \pm\int\sqrt{\alpha_\phi - a(\phi)}\,d\phi$$

Left with 2D eqn:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2\right] + b - E + \frac{2m\alpha_\phi}{r^2\sin^2\theta} = 0$$

Reduced to 2D problem with $V_{eff}(r,\theta) = b(r,\theta) + \frac{2m\alpha_\phi}{r^2\sin^2\theta}$

Compare to Lagrangian:

$$L = \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right) - \frac{a(\phi)}{r^2\sin^2\theta} - b(r,\theta)$$

Eqns: $$m\ddot r - \left(mr\dot\theta^2 + mr\sin^2\theta\,\dot\phi^2 - \frac{\partial b}{\partial r}\right) = 0$$

$$\frac{d}{dt}\left(mr^2\dot\theta\right) - \left(mr^2\sin\theta\cos\theta\,\dot\phi^2 - \frac{\partial b}{\partial \theta}\right) = 0$$

$$\frac{d}{dt}\left(mr^2\sin^2\theta\,\dot\phi\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial a}{\partial \phi} = 0$$

No obvious way to eliminate ϕ

Landau & Lifshitz list seperability criteria for
spherical, parabolic, elliptic coords

So if system is seperable, solution reduced
to integrals.
Usually, integrals can't be done

Can often go further if a system is also periodic

Two flavors: a) p & q periodic fcns of time

System returns to initial state,
like harmonic oscillator

Call cycle = "libration"

b) p is periodic fcn of q

Like pendulum swinging over top
θ keeps increasing, but still
periodic

Call cycle = "rotation"

Can often get period even if not full solution

Use "action - angle" variables

Action variable $J_\sigma \equiv \oint p_\sigma \, dq_\sigma$ over one period

Since separable, know $p_\sigma = \frac{\partial S}{\partial q_\sigma} = \frac{dW_\sigma}{dq_\sigma}$

Remember, seperation consts α_σ also in W

So $p_\sigma = p_\sigma(q_\sigma, \alpha\text{'s})$

$\Rightarrow J_\sigma = J_\sigma(\alpha\text{'s})$

= combination of constants

Assume relation invertable:

$$\alpha_\sigma = \alpha_\sigma(J\text{'s})$$

Express action $S(q\text{'s}, \alpha\text{'s}, t)$ in terms of J's:

$$= \bar{S}(q\text{'s}, J\text{'s}, t)$$

In particular, say α_1 = sep constant for t

So $\bar{S} = \bar{W}(q\text{'s}, J\text{'s}) - \alpha_1(J\text{'s})\, t$

$\bar{S}$ is still action, just with constants rearranged

So $p_\sigma = \left(\frac{\partial \bar{S}}{\partial q_\sigma}\right)_{J\text{'s}}$ $\quad \bar{Q}_\sigma = \left(\frac{\partial \bar{S}}{\partial J_\sigma}\right)_q$ = constant $\equiv \bar{\beta}_\sigma$

$$\bar{\beta}_\sigma = \left(\frac{\partial \bar{W}}{\partial J_\sigma}\right)_q - \frac{\partial \alpha_1}{\partial J_\sigma} t$$

Define "angle variable" $w_\sigma = \left(\frac{\partial \bar{W}}{\partial J_\sigma}\right)_q$

= fcn of q's

Then $w_\sigma = \bar{\beta}_\sigma + \frac{\partial \alpha_1}{\partial J_\sigma} t$

constants

w_σ increases linearly in time
(like angle in const rotation)

Rate $\frac{\partial \alpha_1}{\partial J_\sigma} = \frac{\partial E}{\partial J_\sigma} \equiv \nu_\sigma$

In fact, ν_σ = oscillation freq for mode σ

Show example first, then prove:

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2 = E$$

So $p = \pm \sqrt{2mE - mkq^2}$

$$J = \int p\,dq = \oint \sqrt{2mE - mkq^2}\, dq$$

To integrate set $q = \sqrt{\frac{2E}{k}} \sin\theta$

$$dq = \sqrt{\frac{2E}{k}} \cos\theta$$

$$J = \int_0^{2\pi} \sqrt{2mE(1-\sin^2\theta)} \sqrt{\frac{2E}{k}} \cos\theta\, d\theta$$

$$= 2E\sqrt{\frac{m}{k}} \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= 2E\sqrt{\frac{m}{k}}\, \pi$$

Or $E = \frac{1}{2\pi} J \sqrt{\frac{k}{m}}$

$\nu = \frac{\partial E}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ correct!

Prove in general:

Assume entire system periodic
not just individual coords
→ requires individual freqs to be commensurate

$\Delta t = N_1 \tau_1 = N_2 \tau_2 = \ldots N_n \tau_n$ integers N_σ

If not true, then system arbitrarily close to
one that is (N's can be large)

In time Δt, angle variable w_λ changes by

$$\Delta w_\lambda = \nu_\lambda \Delta t = N_\lambda \nu_\lambda \tau_\lambda \quad *$$

But also know

$$dw_\lambda = \sum_\sigma \frac{\partial w_\lambda}{\partial q_\sigma} dq_\sigma$$

$$= \sum_\sigma \frac{\partial}{\partial q_\sigma}\left(\frac{\partial \overline{W}}{\partial J_\lambda}\right) dq_\sigma$$

$$= \frac{\partial}{\partial J_\lambda} \sum_\sigma \frac{\partial \overline{W}}{\partial q_\sigma} dq_\sigma$$

$$dw_\lambda = \frac{\partial}{\partial J_\lambda} \sum_\sigma p_\sigma dq_\sigma$$

Integrate over time Δt:

$$\Delta w_\lambda = \frac{\partial}{\partial J_\lambda} \sum_\sigma \underbrace{\int_{\Delta t} p_\sigma dq_\sigma}$$

$$= N_\sigma \oint p_\sigma dq_\sigma$$

$$= N_\sigma J_\sigma$$

So $\Delta w_\lambda = \frac{\partial}{\partial J_\lambda} \sum_\sigma N_\sigma J_\sigma = N_\lambda$

Combine with $*$: $N_\lambda \nu_\lambda \tau_\lambda = N_\lambda$

$$\nu_\lambda = \frac{1}{\tau_\lambda} = \text{freq for } \lambda.$$

Useful any time you can express p's as fcns of q's

Harder example: Kepler problem

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{\gamma m}{r}$$

H-J: $\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - \frac{\gamma m}{r} + \frac{\partial S}{\partial t} = 0$

Since ϕ not cyclic, separation is trivial

$$S = W(r) - Et + l\phi$$

$$\frac{1}{2m}\left(\frac{dW}{dr}\right)^2 + \frac{l^2}{2mr^2} - \frac{\gamma m}{r} - E = 0$$

Note $p_\phi = \frac{\partial S}{\partial \phi} = l$

$$p_r = \frac{\partial S}{\partial r} = \frac{dW}{dr} = \pm\left(2mE + \frac{2\gamma m^2}{r} - \frac{l^2}{r^2}\right)^{1/2}$$

Get action variables

$$J_\phi = \oint p_\phi \, d\phi = l \oint d\phi = 2\pi l$$

$$J_r = \pm \oint \sqrt{\quad} \, dr$$

How does r vary over cycle?

Bounded by classical turning points

$$2mE + \frac{2\gamma m^2}{r} - \frac{\ell^2}{r^2} = 0$$

Quadratic in r: roots $r_1 = -\frac{1}{E}\left(\gamma m - \sqrt{\gamma^2 m^2 + \frac{2E\ell^2}{m}}\right)$

$$r_2 = -\frac{1}{E}\left(\gamma m + \sqrt{\gamma^2 m^2 + \frac{2E\ell^2}{m}}\right)$$

Remember $E<0$ ... see $0 < r_1 < r_2$

$$J_r = \oint p_r\,dr = 2\int_{r_1}^{r_2} \sqrt{2mE + \frac{2\gamma m^2}{r} - \frac{\ell^2}{r^2}}\,dr$$

Know that r_1 + r_2 factor root: get

$$J_r = 2\sqrt{-2mE}\int_{r_1}^{r_2} \frac{1}{r}\sqrt{(r_2-r)(r-r_1)}\,dr$$

$$= 2\sqrt{-2mE}\int_{r_1}^{r_2} \frac{1}{r}\sqrt{-r^2+(r_1+r_2)r - r_1 r_2}\,dr$$

Look up formula

$$\int \frac{1}{u}\sqrt{Au^2+Bu+C}\,du = \sqrt{Au^2+Bu+C} - \frac{B}{2\sqrt{-A}}\sin^{-1}\left(\frac{2Au+B}{\sqrt{B^2-4AC}}\right)$$

$$- \sqrt{-C}\,\sin^{-1}\left(\frac{Bu+2C}{|u|\sqrt{B^2-4AC}}\right)$$

for $A, C < 0$

Put in $A=-1$, $B = r_1+r_2$ $C = -r_1 r_2$

$u = r_2$ and r_1

Not so bad...

$$B^2 - 4AC = (r_1 + r_2)^2 - 4r_1 r_2 = (r_1 - r_2)^2$$

$$\sqrt{} = |r_1 - r_2| = r_2 - r_1$$

Eval at r_2: $\sqrt{Au^2 + Bu + C} = 0$

Get $$- \frac{r_1 + r_2}{2} \sin^{-1} \frac{-2r_2 + r_1 + r_2}{r_2 - r_1}$$

$$- \sqrt{r_1 r_2} \sin^{-1} \left(\frac{(r_1 + r_2)(r_2) - 2r_1 r_2}{r_2 (r_2 - r_1)} \right)$$

$$= - \frac{r_1 + r_2}{2} \sin^{-1} \left(\frac{r_1 - r_2}{r_2 - r_1} \right) - \sqrt{r_1 r_2} \sin^{-1} \left(\frac{r_2 (r_2 - r_1)}{r_2 (r_2 - r_1)} \right)$$

$$= - \frac{r_1 + r_2}{2} \left(-\frac{\pi}{2} \right) - \sqrt{r_1 r_2} \left(\frac{\pi}{2} \right)$$

$$= \frac{\pi}{4} \left(r_1 + r_2 - 2\sqrt{r_1 r_2} \right)$$

Eval at r_1 get $-\frac{\pi}{4} \left(r_1 + r_2 - 2\sqrt{r_1 r_2} \right)$

So $$\int_{r_1}^{r_2} = \frac{\pi}{2} \left(r_1 + r_2 - 2\sqrt{r_1 r_2} \right)$$

and $$J_r = \pi \sqrt{-2mE} \left(r_1 + r_2 - 2\sqrt{r_1 r_2} \right)$$

From r_1 & r_2 have

$$r_1 + r_2 = - \frac{2\gamma m}{E}$$

$$r_1 r_2 = - \frac{2\ell^2}{mE}$$

$$J_r = 2\pi \sqrt{-2mE} \left(- \frac{\gamma m}{E} - 2\sqrt{\frac{-\ell^2}{2mE}} \right)$$

$$2\pi \left(\sqrt{\frac{2m^3 \gamma^2}{-E}} - 2|\ell| \right)$$

To use action-angle result, solve for $E(J_r, J_\phi)$

$$J_\phi = 2\pi l$$

so $$J_r = 2\pi\sqrt{\frac{2m^3\gamma^2}{-E}} - 2J_\phi$$

$$J_r + 2J_\phi = 2\pi\left(\frac{2m^3\gamma^2}{-E}\right)^{1/2}$$

$$\boxed{E = -\frac{8\pi^2 m^3\gamma^2}{(J_r + 2J_\phi)^2}}$$

Then frequencies are

$$\nu_\phi = \frac{\partial E}{\partial J_\phi} = \frac{32\pi^2 m^3\gamma^2}{(J_r + 2J_\phi)^3}$$

use $J_r + 2J_\phi = 2\pi\left(\frac{2m^3\gamma^2}{-E}\right)^{1/2}$

so $$\nu_\phi = \frac{32\pi^2 m^3\gamma^2}{8\pi^3 (2m^3\gamma^2)^{3/2}} (-E)^{3/2}$$

$$= \frac{4}{\pi}\frac{1}{2^{3/2}}\frac{1}{\sqrt{m^3\gamma^2}}(-E)^{-3/2}$$

$$\boxed{\nu_\phi = \frac{\sqrt{2}}{\pi}\frac{1}{\gamma m^{3/2}}(-E)^{-3/2}}$$

agrees w/ Ch 1 result ... works!

Similar problem in HW ... $\vec{L}$ in arb direction, not along $\hat{z}$