Lecture 21

Hamilton - Jacobi theory

Have $H(p_\sigma, q_\sigma)$

Canonical transform $p_\sigma \to P_\sigma$, $q_\sigma \to Q_\sigma$

generated by $S(q_\sigma, P_\sigma)$ via

$$Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma}$$

makes $\tilde{H}(P_\sigma, Q_\sigma) = 0$ if

$$H\left(\frac{\partial S}{\partial q_\sigma}, q_\sigma\right) + \frac{\partial S}{\partial t} = 0$$

Solve this PDE to get S

P_σ's = solution constants α_σ

(usually separation consts)

Then equations of motion from $Q_\sigma = \frac{\partial S}{\partial P_\sigma} = \text{const}$

or $\beta_\sigma = \frac{\partial S}{\partial \alpha_\sigma}$

Example: Kepler problem

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{\gamma m}{r}$$

H-J eqn: $$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - \frac{\gamma m}{r} + \frac{\partial S}{\partial t} = 0$$

Since $\frac{\partial H}{\partial t} = \frac{\partial H}{\partial \phi} = 0$, try $S = W(r) - \alpha_1 t + \alpha_2 \phi$

Then $\frac{1}{2m}\left[\left(\frac{dW}{dr}\right)^2 + \frac{\alpha_2^2}{r^2}\right] - \frac{\gamma m}{r} + \alpha_1 = 0$

Recognize $\alpha_1 = E$; $\alpha_2 = l$

$$\left(\frac{dW}{dr}\right)^2 = 2m\alpha_1 + \frac{2\gamma m^2}{r} - \frac{\alpha_2^2}{r^2}$$

$$W(r) = \pm \int \sqrt{2m\alpha_1 + \frac{2\gamma m^2}{r} - \frac{\alpha_2^2}{r}}\, dr + \text{const}$$

Then

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \pm \int \frac{m\,dr}{\sqrt{}}\, dr - t$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \pm \int \frac{\alpha_2}{r} \frac{dr}{\sqrt{}} + \phi$$

β_1 eqn gives $r(t)$

β_2 eqn gives $\phi(r)$

Same integrals we got in Ch 1

Note we didn't need to know $\alpha_1 = E$, $\alpha_2 = l$

Here they are constants that fall out of solution, not tools used to solve problem

Provides an abstract, formal approach to solving any mechanics problem

Effective any time Hamiltonian is separable:

Coordinate q_σ appears only in combination $f(q_\sigma, p_\sigma)$ in H

Pretty general... see Landau & Lifshitz for details

Today, focus on connections to quantum mechanics

Already looking similar:

HJ: $$-\frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial q_\sigma}, q_\sigma, t\right)$$

Schrodinger: $$i\hbar \frac{\partial \psi}{\partial t} = H\left(\frac{\hbar}{i}\frac{\partial}{\partial q_\sigma}, q_\sigma, t\right)\psi$$

for wave function ψ

using quantum prescription $p_\sigma \to \frac{\hbar}{i}\frac{\partial}{\partial q_\sigma}$

To connect, write $\psi(q_\sigma) = A(q_\sigma, t)\, e^{\frac{i}{\hbar}S(q_\sigma, t)}$

$$\frac{\partial \psi}{\partial q_\sigma} = \frac{\partial A}{\partial q_\sigma} e^{\frac{i}{\hbar}S} + \frac{i}{\hbar}\frac{\partial S}{\partial q_\sigma} A e^{\frac{i}{\hbar}S}$$

In classical limit $\hbar \to 0$
2nd term dominates

Then $$\frac{\partial \psi}{\partial q_\sigma} \to \frac{i}{\hbar}\frac{\partial S}{\partial q_\sigma}\psi$$

or $$p_\sigma \psi = \frac{\hbar}{i}\frac{\partial \psi}{\partial q_\sigma} = \frac{\partial S}{\partial q_\sigma}\psi$$

So Schrödinger becomes

$$i\hbar \frac{\partial \psi}{\partial t} \approx H\left(\frac{\partial S}{\partial q_\sigma}, q_\sigma, t\right)\psi$$

Similarly, $$\frac{\partial \psi}{\partial t} = \frac{\partial A}{\partial t} e^{\frac{i}{\hbar}S} + \frac{i}{\hbar}\frac{\partial S}{\partial t} A e^{\frac{i}{\hbar}S} \approx \frac{i}{\hbar}\frac{\partial S}{\partial t}\psi$$

So $$i\hbar\frac{\partial \psi}{\partial t} \approx (i\hbar)\left(\frac{i}{\hbar}\frac{\partial S}{\partial t}\psi\right) = -\frac{\partial S}{\partial t}\psi$$

So in limit $\hbar \to 0$, Schr becomes

$$-\frac{\partial S}{\partial t}\psi = H\left(\frac{\partial S}{\partial q_\sigma}, q_\sigma, t\right)\psi$$

Cancel ψ's, left with H-J eqn

See that classical physics emerges from QM
when phase of wave function is rapidly varying

/ Used in QM itself: WKB approximation

Similar in optics wave eqn $\to$ ray optics

So S = generating function for transform $\to$ phase of ψ

More about S:

$$S = S(q_\sigma, P_\sigma, t) \qquad P_\sigma = \text{const}$$

$$\frac{dS}{dt} = \sum_\sigma \frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial t}$$

but $\left(\frac{\partial S}{\partial q_\sigma}\right) = p_\sigma$

and $\frac{\partial S}{\partial t} = -H$ since $\tilde{H} = H + \frac{\partial S}{\partial t} = 0$

So $\frac{dS}{dt} = \sum_\sigma p_\sigma \dot{q}_\sigma - H = L$ Lagrangian

Or,

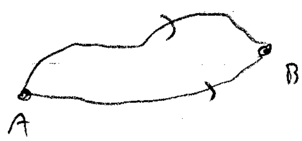
$$S = \int L\, dt$$

$= \underline{\text{action}}$!

So classical action → phase of wave function

Used in path-integral formulation of QM

Quantum particle
takes all paths
from A to B



Each path has phase $\phi = \frac{1}{\hbar} \times$ action

If S varies rapidly, amplitudes for nearby paths cancel
→ low probability to reach B

If S stationary, nearby paths add up

⇒ large prob to find particle on path
where $\delta S = 0$
⇒ Hamilton's principle

So that explains how classical theory develops
from QM

How do we make quantum theory from classical one?

Use Poisson brackets

Suppose $F(q_\sigma, p_\sigma, t)$, $G(q_\sigma, p_\sigma, t)$ arb functions

Define
$$[F, G]_{PB} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right)$$

= another fcn of q's & p's

Consider $[H, F]_{PB}$ for Hamiltonian H

$$[H,F]_{PB} = \sum_\sigma \frac{\partial H}{\partial q_\sigma}\frac{\partial F}{\partial p_\sigma} - \frac{\partial H}{\partial p_\sigma}\frac{\partial F}{\partial q_\sigma}$$

$$= \sum_\sigma (-\dot{p}_\sigma)\frac{\partial F}{\partial p_\sigma} - \dot{q}_\sigma \frac{\partial F}{\partial q}$$

$$= -\left(\frac{dF}{dt} - \frac{\partial F}{\partial t}\right)$$

or $$\boxed{\frac{dF}{dt} = -[H,F]_{PB} + \frac{\partial F}{\partial t}}$$

eqn of motion for any F

Example: $H = \frac{p^2}{2m} + mgz$

$F = z$: $$[H,z]_{PB} = \frac{\partial H}{\partial z}\frac{\partial z}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial z}{\partial z}$$

$$= 0 - \frac{p}{m} \times 1$$

So $$\frac{dz}{dt} = -[H,z] = \frac{p}{m} \quad ✓$$

$F = p$. $$[H,p] = \frac{\partial H}{\partial z}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial z}$$

$$= mg - 0$$

$$\frac{dp}{dt} = -[H,p] = -mg$$

If we wanted $F = \frac{p^2 z}{2z+1}$, could get time derivative

Note if $\frac{\partial F}{\partial t} = 0$ & $(H,F) = 0$, then $F = \text{const}$

For instance, $[H,H] = 0$ always, so $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ ✓

Don't need to use H:

$$[q_\alpha, p_\beta] = \sum_\sigma \frac{\partial q_\alpha}{\partial q_\sigma}\frac{\partial p_\beta}{\partial p_\sigma} - \frac{\partial q_\alpha}{\partial p_\sigma}\frac{\partial p_\beta}{\partial q_\sigma}$$

$$= \sum_\sigma (\delta_{\alpha\sigma}\delta_{\beta\sigma} - 0) = \delta_{\alpha\beta}$$

$$= -[p_\alpha, q_\beta]$$

Also $[q_\alpha, q_\beta] = [p_\alpha, p_\beta] = 0$

These relations hold whenever q + p are conjugates

$\Rightarrow$ Way to check if transformation $(p,q) \to (P,Q)$ is canonical:

See if $[Q_\alpha, P_\beta] = \delta_{\alpha\beta}$

Consider $[F, q_\beta]$, arb F

$$= \sum_\sigma \frac{\partial F}{\partial q_\sigma}\frac{\partial q_\beta}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial q_\beta}{\partial q_\sigma}$$

$$= \sum_\sigma \left(0 - \frac{\partial F}{\partial p_\sigma}\delta_{\beta\sigma}\right)$$

$$[F, q_\beta]_{PB} = -\frac{\partial F}{\partial p_\beta}$$

also $[F, p_\beta]_{PB} = +\frac{\partial F}{\partial q_\beta}$

Recognize Poisson brackets $\approx$ commutators in QM

$[F, G] \equiv FG - GF$ Operators F, G

Operator: fcn $\to$ fcn

Correct prescription to go from classical to quantum:

1. Interpret all classical variables as operators

2. Replace $[F,G]_{PB} \rightarrow \frac{i}{\hbar}[F,G]$

$\Rightarrow$ Theory of operators acting on abstract space
= Hilbert space

In particular:

$$[p_\alpha, q_\beta]_{PB} \rightarrow \frac{1}{i\hbar}[p_\alpha, q_\beta] = -\delta_{\alpha\beta}$$

$$\text{so} \quad [p_\alpha, q_\beta] = -i\hbar\,\delta_{\alpha\beta}$$

Source of Heisenberg uncertainty principle

See it holds for any canonical conjugates

Can derive all of QM from this

For instance, eqn of motion

$$\frac{dF}{dt} = \frac{i}{\hbar}[H,F] + \frac{\partial F}{\partial t}$$

Suppose operator acts on "state" $|\psi\rangle$

Define "density operator" $|\psi\rangle\langle\psi| = F$

for state with $\langle\psi|\psi\rangle = 1$ and $\frac{d}{dt}|\psi\rangle = 0$

Then $\frac{dF}{dt} = 0 = \frac{i}{\hbar}\left[H, |\psi\rangle\langle\psi|\right] + \frac{\partial}{\partial t}|\psi\rangle\langle\psi|$

$$H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H = i\hbar\left[\left|\frac{\partial\psi}{\partial t}\right\rangle\langle\psi| + |\psi\rangle\left\langle\frac{\partial\psi}{\partial t}\right|\right]$$

apply this to $|\psi\rangle$, & use $\langle\psi|\psi\rangle = 1$:

$$H|\psi\rangle - |\psi\rangle\langle\psi|H|\psi\rangle = i\hbar\left[\left|\frac{\partial\psi}{\partial t}\right\rangle + |\psi\rangle\left\langle\frac{\partial\psi}{\partial t}\middle|\psi\right\rangle\right]$$

But we know $\frac{\partial}{\partial t}\langle\psi|\psi\rangle = \left\langle\frac{\partial\psi}{\partial t}\middle|\psi\right\rangle + \left\langle\psi\middle|\frac{\partial\psi}{\partial t}\right\rangle = 0$

so $\left\langle\frac{\partial\psi}{\partial t}\middle|\psi\right\rangle = -\left\langle\psi\middle|\frac{\partial\psi}{\partial t}\right\rangle$

$$H|\psi\rangle - |\psi\rangle\langle\psi|H|\psi\rangle = i\hbar\left[\left|\frac{\partial\psi}{\partial t}\right\rangle - |\psi\rangle\left\langle\psi\middle|\frac{\partial\psi}{\partial t}\right\rangle\right]$$

or $\left(1 - |\psi\rangle\langle\psi|\right)H|\psi\rangle = i\hbar\left(1 - |\psi\rangle\langle\psi|\right)\left|\frac{\partial\psi}{\partial t}\right\rangle$

assuming $|\psi\rangle\langle\psi| \neq I$, get

$$\boxed{H|\psi\rangle = i\hbar\left|\frac{\partial\psi}{\partial t}\right\rangle}$$ Schr. Eqn

Further, consider $[p, F]_{PB} = -\frac{\partial F}{\partial q}$

$$\rightarrow [p, F] = -i\hbar\frac{\partial F}{\partial q}$$

again with $F = |\psi\rangle\langle\psi|$

$$p|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|p = -i\hbar\left(\left|\frac{\partial\psi}{\partial q}\right\rangle\langle\psi| + |\psi\rangle\left\langle\frac{\partial\psi}{\partial q}\right|\right)$$

Apply to $|\psi\rangle$:

$$p|\psi\rangle - |\psi\rangle\langle\psi|p|\psi\rangle = -i\hbar\left(\left|\frac{\partial\psi}{\partial q}\right\rangle + |\psi\rangle\left\langle\frac{\partial\psi}{\partial q}\middle|\psi\right\rangle\right)$$

again $\left\langle\psi\middle|\frac{\partial\psi}{\partial q}\right\rangle = -\left\langle\frac{\partial\psi}{\partial q}\middle|\psi\right\rangle$

$$(1 - |\psi\rangle\langle\psi|)\, p|\psi\rangle = -i\hbar (1 - |\psi\rangle\langle\psi|)\, |\tfrac{\partial\psi}{\partial q}\rangle$$

$$p|\psi\rangle = -i\hbar\, |\tfrac{\partial\psi}{\partial q}\rangle$$

So if $H = \frac{p^2}{2m} + V(q)$

$$H|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$$

becomes $$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V\right)|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

So: Classical $\to$ quantum: $[\ ]_{PB} \to -\frac{i}{\hbar}[\ ]$

Quantum $\to$ classical: $\psi \to A e^{\frac{i}{\hbar}S}$

S rapidly varying

Complete connection between theories