Lecture 20

Hamitonian dynamics:

Use variables (q_σ, p_σ)

Eqns of motion
$$\dot{q}_\sigma = \left(\frac{\partial H}{\partial p_\sigma}\right)_q$$
$$\dot{p}_\sigma = -\left(\frac{\partial H}{\partial q_\sigma}\right)_p$$

Lagrangian dynamics allows "point transformations"
$$q_\sigma \to Q_\sigma(q's, t)$$

Preserves eqn of motion $\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_\sigma} - \frac{\partial L}{\partial Q_\sigma} = 0$

Consider more general transform
$$q_\sigma \to Q_\sigma(q's, p's, t)$$
$$p_\sigma \to P_\sigma(q's, p's, t)$$

Assume invertible, so
$$q_\sigma = q_\sigma(Q, P, t)$$
$$p_\sigma = p_\sigma(Q, P, t)$$

Also require eqn of motion unchanged:
$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

for some $\tilde{H}$

Use Hamilton's Principle $\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0$
$$L = \sum p_\sigma \dot{q}_\sigma - H$$

so
$$\delta \int \left(\sum_\sigma p_\sigma \dot{q}_\sigma - H\right) dt = 0$$

Require new variables to satisfy
$$\delta \int \left(\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\right) dt = 0$$

Works if
$$\boxed{\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H} = \sum p_\sigma \dot{q}_\sigma - H + \frac{dF}{dt}}$$

arb function F

For instance, might express $F = f(q, p, t)$

then $\delta \int \frac{dF}{dt}\, dt = \delta \left[f(q_2, p_2, t_2) - f(q_1, p_1, t_1) \right]$

$= 0$ since initial & final conditions fixed

This defines canonical transform

Rather awkward definition...

Given $Q(q, p, t)$ and $P(q, p, t)$ how do we know if it is canonical or not?

Turns out, Poisson brackets provide a test [see later

For now, reverse question...

Given $\tilde{H}$ and F, what transform is defined?

In fact, only need F:

expressed as function of both old and new coords

$\to$ any function of (q, Q), (q, P), (p, Q) or (p, P)

call F = generating function

Suppose $F(q, Q, t)$

Then $$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial Q_\sigma} \dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

Canonical condition becomes

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(p, q) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(P, Q) + \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial Q_\sigma} \dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

or $$\sum_\sigma \left(p_\sigma - \frac{\partial F}{\partial q_\sigma} \right) \dot{q}_\sigma - H = \sum_\sigma \left(P_\sigma - \frac{\partial F}{\partial Q_\sigma} \right) \dot{Q}_\sigma - \tilde{H} + \frac{\partial F}{\partial t}$$

Satisfied if

a) $$\boxed{P_\sigma = -\frac{\partial}{\partial Q_\sigma} F(q, Q)}$$

$\Rightarrow$ relation between P's, Q's & q's

Implicitly gives $q_\sigma(P, Q, t)$

b) $$\boxed{p_\sigma = \frac{\partial}{\partial q_\sigma} F(q, Q)}$$

$\Rightarrow$ relation between p's, q's & Q's

$\Rightarrow$ using (a), relation between p', Q's, P's

Implicitly gives $p_\sigma(P, Q, t)$

c) Given $q(P, Q)$ and $p(P, Q)$

invert to get $Q(p, q)$, $P(p, q)$

$\Rightarrow$ Transformation implicitly defined

c) $$\boxed{\tilde{H} = H + \frac{\partial F}{\partial t}}$$ defines new Hamiltonian,
Express in terms of (P, Q)

Examples:

$$F = \sum_\sigma q_\sigma Q_\sigma$$

$$p_\sigma = \frac{\partial F}{\partial q_\sigma} = Q_\sigma$$

$$P_\sigma = -\frac{\partial F}{\partial Q_\sigma} = -q_\sigma$$

or $(q, p) \rightarrow (p, -q)$

Exchange roles of coords & momenta

Here $\tilde{H} = H$ since $\frac{\partial F}{\partial t} = 0$

Example: $F = \frac{1}{2\alpha} \sum_\sigma (q_\sigma - Q_\sigma)^2$

$$\frac{\partial F}{\partial q_\sigma} = \frac{1}{\alpha}(q_\sigma - Q_\sigma) = p_\sigma$$

$$\Rightarrow \boxed{Q_\sigma = q_\sigma - \alpha p_\sigma}$$

and $\frac{\partial F}{\partial Q_\sigma} = -\frac{1}{\alpha}(q_\sigma - Q_\sigma) = -P_\sigma$

$$\alpha P_\sigma = q_\sigma - Q_\sigma = q_\sigma - (q_\sigma - \alpha p_\sigma)$$
$$= \alpha p_\sigma$$

So $\boxed{P_\sigma = p_\sigma}$

Again $\tilde{H} = H$

So, any $F(q, Q, t)$ defines transform

Turns out, function of (q, P, t) often most convenient

Changes generating relations...

Change from (q, Q) to (q, P) where $P = -\frac{\partial F}{\partial Q}$

$\Rightarrow$ Legendre transform

Define $S(q; P, t) = F(q, Q; t) + \sum_\sigma P_\sigma Q_\sigma$

Then know $\left(\frac{\partial S}{\partial q_\sigma}\right)_P = \left(\frac{\partial F}{\partial q_\sigma}\right)_Q = p_\sigma$

by property of Legendre

Also, clearly have $\frac{\partial S}{\partial P_\sigma} = Q_\sigma$ (since $\frac{\partial F}{\partial P_\sigma} = 0$)

So now $\boxed{p_\sigma = \left(\frac{\partial S}{\partial q_\sigma}\right)_P, \quad Q_\sigma = \left(\frac{\partial S}{\partial P_\sigma}\right)_q}$

Example:

$$S(q,P,t) = \sum_\sigma f_\sigma(q,t)\, P_\sigma$$

$$Q_\sigma = \frac{\partial S}{\partial P_\sigma} = f(q,t)$$

Defines point-transform (what Lagrangian allows)

So any point transform is a canonical transform

Also set $\quad p_\sigma = \dfrac{\partial S}{\partial q_\sigma} = \displaystyle\sum_\lambda \frac{\partial f_\lambda}{\partial q_\sigma} P_\lambda$

invert to get P_λ

Equivalent to finding $L(Q,\dot{Q})$ and setting

$$P_\sigma = \frac{\partial L}{\partial \dot{Q}_\sigma}$$

Example:

$$S_0 = \sum_\sigma q_\sigma P_\sigma$$

$$\left.\begin{aligned} p_\sigma &= \frac{\partial S}{\partial q_\sigma} = P_\sigma \\ Q_\sigma &= \frac{\partial S}{\partial P_\sigma} = q_\sigma \end{aligned}\right]$$

Identity transform... does nothing

Example:

$S = S_0 + H\,dt \qquad H =$ original Hamiltonian

$dt =$ small parameter

$$p_\sigma = \left(\frac{\partial S}{\partial q_\sigma}\right)_{\underline{P}} = \left(\frac{\partial S_0}{\partial q_\sigma}\right)_{\underline{P}} + \left(\frac{\partial H}{\partial q_\sigma}\right)_{\underline{P}} dt$$

$$= P_\sigma + \left(\frac{\partial H}{\partial q_\sigma}\right)_{\underline{P}} dt$$

We know $\left(\dfrac{\partial H}{\partial q_\sigma}\right)_p = -\dot{p}_\sigma \quad \ldots \quad$ what is $\left(\dfrac{\partial H}{\partial q_\sigma}\right)_{\underline{P}}$?

Chain rule formula:

$$\left(\frac{\partial H}{\partial q_\sigma}\right)_{\underline{P}} = \left(\frac{\partial H}{\partial q_\sigma}\right)_p + \sum_\lambda \left(\frac{\partial H}{\partial p_\lambda}\right)_q \left(\frac{\partial p_\lambda}{\partial q_\sigma}\right)_{\underline{P}}$$

$$= -\dot{p}_\sigma + \sum_\lambda \dot{q}_\lambda \left(\frac{\partial p_\lambda}{\partial q_\sigma}\right)_{\underline{P}}$$

Need $\left(\frac{\partial p_\lambda}{\partial q_\sigma}\right)_{\underline{P}}$

Use $p_\lambda = P_\lambda + \left(\frac{\partial H}{\partial q_\lambda}\right)_{\underline{P}} dt$

$$\left(\frac{\partial p_\lambda}{\partial q_\sigma}\right)_{\underline{P}} = \left(\frac{\partial^2 H}{\partial q_\lambda \partial q_\sigma}\right)_{\underline{P}} dt$$

So $\left(\frac{\partial H}{\partial q_\sigma}\right)_{\underline{P}} = -\dot{p}_\sigma + \sum_\lambda \dot{q}_\lambda \left(\frac{\partial^2 H}{\partial q_\lambda \partial q_\sigma}\right)_{\underline{P}} dt$

and

$$p_\sigma = P_\sigma - \dot{p}_\sigma dt + \sum_\lambda \dot{q}_\lambda \left(\frac{\partial^2 H}{\partial q_\lambda \partial q_\sigma}\right) dt^2$$

In limit of small dt, last term $\to 0$

Leaves $\underline{P}_\sigma(t) = p_\sigma(t) + \dot{p}_\sigma(t)\, dt$

$$\boxed{P_\sigma(t) = p_\sigma(t+dt)}$$

Similarly, get $\boxed{Q_\sigma = q_\sigma(t+dt)} + O(dt^2)$

So this <u>S generates propagation in time!</u>

In HW, show that translations & rotations are canonical transforms too.

Coolest transform: make $\tilde{H} = 0$

Then
$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} = 0$$
$$\dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma} = 0$$

All Q's & P's = constants

From $q_\sigma = q_\sigma(Q, P, t)$, motion solved

But finding transform isn't easy

Want $\tilde{H}(P, Q, t) = H(p, q, t) + \frac{\partial S}{\partial t} = 0$

Have $p_\sigma = \frac{\partial S}{\partial q_\sigma}$, so

$$\boxed{H\left(\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots \frac{\partial S}{\partial q_n}, q_1, q_2, \dots q_n, t\right) + \frac{\partial S}{\partial t} = 0}$$

ie, write out $H(p,q)$, replace $p \to \frac{\partial S}{\partial q}$ everywhere

Example: $H = \frac{p^2}{2m} + V(q)$

gives $\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) + \frac{\partial S}{\partial t} = 0$

PDE for S

Called Hamilton-Jacobi equation

Solution gives $S(q,t)$ ... what about $\underline{P}$'s?

$\underline{P}$'s are integration constants in solution

H-J eqn has $n+1$ variables! $q_1, q_2, \dots q_n, t$

→ Has $n+1$ constants in solution
get from initial conditions

n of these are P_σ's

Last one is arb. additive const $S \to S + \text{const}$

(only derivs in H-J eqn)

Example $H = \frac{p^2}{2m} + V(q)$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) + \frac{\partial S}{\partial t} = 0$$

Note, t doesn't appear explicitly...

Try solution $S(q,t) = W(q) - \alpha_1 t$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + U(q) - \alpha_1 = 0$$

time independent

Already significant:

$$\tilde{H} = H + \frac{\partial S}{\partial t} = 0$$

$$\text{So } H = -\frac{\partial S}{\partial t} = \alpha_1$$

Proves $H = \text{const}$... knew already

Also shows $\alpha_1 = E$

Then $\frac{dW}{dq} = \pm\sqrt{2m(E-V)}$

$$W(q) = \underbrace{\pm \int \sqrt{2m(E-V)}\, dq}_{\text{indefinite integral}} + \underbrace{S_0}_{\text{overall constant, irrelevant}}$$

$\pm$ sign set by initial conditions

Given $V(q)$, could maybe do integral

anyway, have formal solution

Use S to get solution

Picked S so P_σ & Q_σ = const

Already identified $P_\sigma = \alpha_\sigma$

What about Q_σ?

$$Q_\sigma = \frac{\partial S}{\partial P_\sigma} \equiv \beta_\sigma = \text{const}$$

So $\beta_\sigma = \dfrac{\partial S}{\partial \alpha_\sigma} = \text{const}$ } Set of n eqns involving q_σ's and t

Solve to get $q_\sigma(t)$

Example:

$$S = \pm \int \sqrt{2m(E-V)}\, dq - Et$$

one coord, $\alpha_1 = E$

So $\beta = \dfrac{\partial S}{\partial E} = \pm \int \sqrt{2m}\, \frac{1}{2} \frac{1}{\sqrt{E-V}}\, dq - t$

$$\boxed{t = \beta \pm \sqrt{\frac{m}{2}} \int \frac{dq}{\sqrt{E-V}}}$$

Same thing we get by solving $\frac{1}{2} m\dot{q}^2 + V(q) = E$

$$\dot{q} = \pm\sqrt{\frac{2}{m}(E-V)}$$

$$\pm \int \sqrt{\frac{m}{2}} \frac{dq}{\sqrt{E-V}} = \int dt = t + \text{const}$$

See β = another integration constant

Another example: Kepler problem

Cylindrical coords (r, ϕ)

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{\gamma m}{r}$$

So $$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{\gamma m}{r} + \frac{\partial S}{\partial t} = 0$$

Since $\frac{\partial H}{\partial t} = \frac{\partial H}{\partial \phi} = 0$, try

$$S(r, \phi, t) = W(r) - \alpha_1 t + \alpha_2 \phi$$

Then $$\frac{1}{2m}\left[\left(\frac{dW}{dr}\right)^2 + \frac{\alpha_2^2}{r^2}\right] - \frac{\gamma m}{r} + \alpha_1 = 0$$

(recognize $\alpha_1 = E \quad \alpha_2 = L$)

$$\left(\frac{dW}{dr}\right)^2 = 2m\alpha_1 + \frac{2\gamma m^2}{r} - \frac{\alpha_2^2}{r^2}$$

$$W = \pm \int \sqrt{2m\alpha_1 + \frac{2\gamma m}{r} - \frac{\alpha_2^2}{r^2}}\, dr$$

Then $$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \pm \int \frac{m\,dr}{\sqrt{2m\alpha_1 + \frac{2\gamma m}{r} - \frac{\alpha_2^2}{r^2}}}\, dr - t$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \pm \int \frac{\alpha_2}{r} \frac{dr}{\sqrt{\quad}} + \phi$$

Solve β_1 eqn to get $r(t)$

β_2 to get $\phi(r)$ or $r(\phi)$

Exactly same integrals we got in Ch 1

But notice: didn't need to know $\alpha_1 = E$, $\alpha_2 = L$ = const

Here they are const that fall out of solution,

not tools used to solve problem