Lecture 19

Start new topic: Hamiltonian dynamics

= another formulation of mechanics ... Why?

Already have two ways
Newtonian: direct & intuitive $(\vec{r}, \dot{\vec{r}})$
Lagrangian: useful for constraints $(q_\sigma, \dot{q}_\sigma)$

Hamiltonian much like Lagrangian, but use (q_σ, p_σ)
rather than $(q_\sigma, \dot{q}_\sigma)$

Several advantages:
(1) Often have $p_\sigma = \text{const}$ ... convenient

(2) More flexibility in coords

Lagrange: $q_\sigma \to Q_\sigma = f_\sigma(q\text{'s}, t)$

$$\dot{q}_\sigma \to \dot{Q}_\sigma = \frac{df_\sigma}{dt}$$

Hamilton: $q_\sigma \to f_\sigma(q\text{'s}, p\text{'s}, t)$

$$p_\sigma \to g_\sigma(q\text{'s}, p\text{'s}, t)$$

certain restrictions on f & g

Class of transformations = "Canonical transforms"

Permits some fancy tricks
ie, find coords where every coord = constant

- Not so useful for practical solutions, but nice for theoretical derivations

(3) Close relation to QM
- Can make Hamiltonian formalism almost exactly like quantum theory

$\Rightarrow$ hope to understand QM better

Start by going from $(q, \dot{q})$ to (q, p)

<u>Example</u> Particle in plane, potential $U(r, \phi)$

$$L = T - V = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - U(r, \phi)$$

Eqns of motion $\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m \dot{r} \qquad \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$\frac{\partial L}{\partial r} = m r \dot{\phi}^2 - \frac{\partial U}{\partial r} \qquad \frac{\partial L}{\partial \phi} = - \frac{\partial U}{\partial \phi}$$

Recall $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$

So $p_r = m \dot{r}$ $\qquad p_\phi = m r^2 \dot{\phi}$

Eqns of motion $\rightarrow$

$$\frac{dp_r}{dt} = \frac{\partial L}{\partial r} = m r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{dp_\phi}{dt} = \frac{\partial L}{\partial \phi} = - \frac{\partial U}{\partial \phi}$$

Put RHS in terms of p's:

$$mr\dot{\phi}^2 = \frac{(mr^2\dot{\phi})^2}{mr^3} = \frac{p_\phi^2}{mr^3}$$

So
$$\boxed{\begin{aligned}\dot{p}_r &= \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}\\ \dot{p}_\phi &= -\frac{\partial V}{\partial \phi}\end{aligned}}$$
new eqns of motion for (r, p_r, ϕ, p_ϕ)

Got from eqns for $(r, \dot{r}, \phi, \dot{\phi})$

Say we tried to skip that step...

write $L(r, p_r, \phi, p_\phi)$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - U(r,\phi)$$

Try $\dot{p}_r = \frac{\partial L}{\partial r}$ as before

$$\dot{p}_r = -\frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r} \quad \times \quad \text{Wrong!}$$

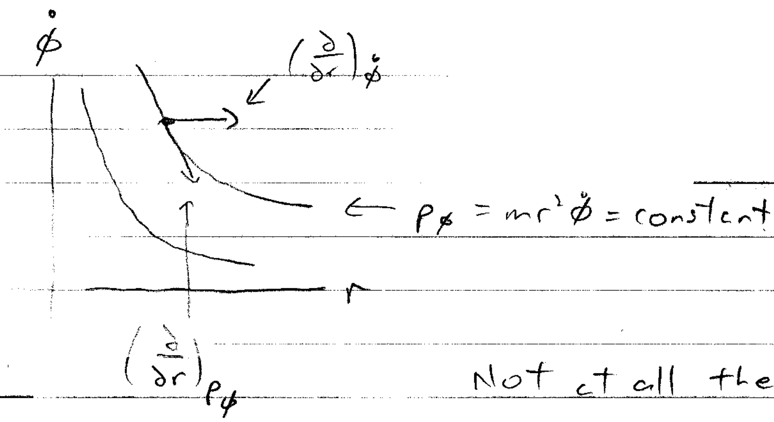
What's the problem?

Lagrange eqns really $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)_{r,\phi,\dot{\phi}} = \left(\frac{\partial L}{\partial r}\right)_{\dot{r},\phi,\dot{\phi}}$

fixed

or $\dot{p}_r = \left(\frac{\partial L}{\partial r}\right)_{\phi,\dot{r},\dot{\phi}} \neq \left(\frac{\partial L}{\partial r}\right)_{\phi,p_r,p_\phi}$

$\left(\frac{\partial L}{\partial r}\right)_{\dot{r}}$ and $\left(\frac{\partial L}{\partial r}\right)_{p_r}$ are derivatives in different directions



Not at all the same

Relate using chain rule:

Say $f = f(x, y)$ and $y = y(x, z)$

so $f = f(x, y(x,z))$

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_z = \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z}$$

Chain rule for partial derivatives

Applying here, get

$$\left(\frac{\partial L}{\partial r}\right)_{p_\phi} = \left(\frac{\partial L}{\partial r}\right)_{\dot\phi} + \left(\frac{\partial L}{\partial \dot\phi}\right)_r \left(\frac{\partial \dot\phi}{\partial r}\right)_{p_\phi} \qquad \dot\phi = \frac{p_\phi}{mr^2}$$

$$-\frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r} = \left(\frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}\right) + p_\phi\left(-\frac{2p_\phi}{mr^3}\right)$$

works ✓

That's basic problem:

Lagrange has $\dot p_\sigma = \left(\frac{\partial L}{\partial q_\sigma}\right)_{\dot q\text{'s}}$

We'd like $\dot p_\sigma = \left(\frac{\partial \text{"something"}}{\partial q_\sigma}\right)_{p_\sigma\text{'s}}$

Turns out "something" is Hamiltonian H

$$H = \sum_\lambda p_\lambda \dot{q}_\lambda - L$$

$$\left(\frac{\partial H}{\partial q_\sigma}\right)_{p's} = \sum_\lambda p_\lambda \left(\frac{\partial \dot{q}_\lambda}{\partial q_\sigma}\right)_p - \left(\frac{\partial L}{\partial q_\sigma}\right)_p$$

$$= \sum_\lambda p_\lambda \left(\frac{\partial \dot{q}_\lambda}{\partial q_\sigma}\right)_p - \left[\left(\frac{\partial L}{\partial q_\sigma}\right)_{\dot{q}} + \sum_\lambda \left(\frac{\partial L}{\partial \dot{q}_\lambda}\right)_q \left(\frac{\partial \dot{q}_\lambda}{\partial q_\sigma}\right)_p\right]$$

$$= \sum_\lambda p_\lambda \left(\frac{\partial \dot{q}_\lambda}{\partial q_\sigma}\right)_p - \left(\frac{\partial L}{\partial q_\sigma}\right)_{\dot{q}} - \sum_\lambda p_\lambda \left(\frac{\partial \dot{q}_\lambda}{\partial q_\sigma}\right)_p$$

$$= -\left(\frac{\partial L}{\partial q_\sigma}\right)_{\dot{q}} = -\dot{p}_\sigma \quad \checkmark$$

Could have defined $H = L - p\dot{q}$ to avoid minus sign, but then $H = -E$, less nice

Example of Legendre Transform:

Given function $f(x, y)$ with $z = \left(\frac{\partial f}{\partial y}\right)_x$

Make new function $g(x, z) = f - yz$

Then $\left(\frac{\partial g}{\partial x}\right)_z = \left(\frac{\partial f}{\partial x}\right)_y$

Change variable from y to z w/o affecting x-derivatives

Very important in Stat Mech... see next semester

$\Rightarrow$ Stat mech classes??

So for Hamiltonian formulation, eqn of motion is

$$\boxed{\dot{p}_\sigma = -\left(\frac{\partial H}{\partial q_\sigma}\right)_p}$$

Can also work out

$$\left(\frac{\partial H}{\partial p_\sigma}\right)_q = \left(\frac{\partial}{\partial p_\sigma}\left[\sum_\lambda p_\lambda \dot{q}_\lambda - L\right]\right)_q$$

$$= \dot{q}_\sigma + \sum_\lambda p_\lambda \left(\frac{\partial \dot{q}_\lambda}{\partial p_\sigma}\right)_q - \left(\frac{\partial L}{\partial p_\sigma}\right)_q$$

$$= \dot{q}_\sigma + \sum_\lambda p_\lambda \left(\frac{\partial \dot{q}_\lambda}{\partial p_\sigma}\right)_q - \sum_\lambda \underbrace{\left(\frac{\partial L}{\partial \dot{q}_\lambda}\right)_q}_{p_\lambda} \left(\frac{\partial \dot{q}_\lambda}{\partial p_\sigma}\right)_q$$

$$= \dot{q}_\sigma$$

So $\boxed{\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}}$ usually just reproduces $p = p(\dot{q})$ function

Two coupled 1st-order eqns
instead of one 2nd order eqn

- Note, in general, still need to calculate L in order to get H to start with

But for time-indep constraints, have $H = T + V$
Can get p from

$$p_\sigma = \frac{\partial T}{\partial \dot{q}_\sigma}$$ since V indep $\dot{q}_\sigma$

Example: Simple harmonic oscillator

$$T = \tfrac{1}{2} m \dot{x}^2 \qquad U = \tfrac{1}{2} m \omega^2 x^2$$

$$p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

$$H = T + V = \frac{p^2}{2m} + \tfrac{1}{2} m \omega^2 x^2$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \quad \Rightarrow \quad p = m\dot{x}, \text{ already had}$$

$$\frac{\partial H}{\partial x} = m\omega^2 x = -\dot{p}$$

$$\hookrightarrow -m\ddot{x}$$

Gives $\ddot{x} = -\omega^2 x$, as expected

Could also solve $\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix}$

directly, by diagonalizing matrix

Example: Top in gravity

$$L = \tfrac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \tfrac{1}{2} I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})^2 - Mg\,l\cos\beta$$

$$p_\alpha = \frac{\partial T}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2\beta + I_3 \cos\beta (\dot{\alpha}\cos\beta + \dot{\gamma})$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})$$

Gives $H = T + V = \dfrac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + \dfrac{p_\beta^2}{2I_1} + \dfrac{p_\gamma^2}{2I_3} + Mgl\cos\beta$

$$= \frac{p_\beta^2}{2I_1} + V_{eff}(p_\alpha, p_\gamma, \beta)$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial \alpha} = 0$$

$$\dot{p}_\beta = -\frac{\partial H}{\partial \beta} = -\left(\frac{\partial V_{eff}}{\partial \beta}\right)_{p_\alpha, p_\gamma}$$

$$\dot{p}_\gamma = -\frac{\partial H}{\partial \gamma} = 0$$

We basically did it this way in Ch 5 without introducing formalism

Can get "standard" H's for common coords

Cartesian $H = \dfrac{p_x^2 + p_y^2 + p_z^2}{2m} + U(x, y, z)$

Cylindrical $H = \dfrac{1}{2m}\left(p_r^2 + p_z^2 + \dfrac{1}{r^2} p_\phi^2\right) + V(r, \phi, z)$

Spherical $H = \dfrac{1}{2m}\left(p_r^2 + \dfrac{1}{r^2} p_\theta^2 + \dfrac{1}{r^2 \sin^2\theta} p_\phi^2\right) + U(r, \theta, \phi)$

When is it convenient to use H instead of L?

When symmetry suggest one or more p's = constant

Terminology: if $\dfrac{\partial L}{\partial q_\sigma} = 0$, say q_σ is cyclic

Then also $\dfrac{\partial H}{\partial q_\sigma} = 0$

Two consequences

1) p_σ = constant

2) $\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$ is indep of q_σ

$= f(q_1 \ldots q_{\sigma-1}, q_{\sigma+1}, \ldots p's, t)$

Can write formal solution

$$q_\sigma(t) = \int f(\ldots)\,dt$$

eliminate q_σ from problem

So cyclic coords really "drop out"
effectively reduce # of degrees of freedom

Saw already in central force + spinning top problems

→ if only one non-cyclic coordinate, problem reduces to 1D motion

= solved

Get same result in Lagrangian formalism, but more direct here

True power of formalism not really in solving problems directly, but in exploiting flexibility of transformations... next time.