Lecture 18

Continuing with Euler angles

First, got a little confused about signs last time, but math was fine

Relate inertial and body coords using

$$\mathbb{R}_{body \leftarrow inert} = \begin{bmatrix} c(\gamma) & s(\gamma) & 0 \\ -s(\gamma) & c(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\beta) & 0 & -s(\beta) \\ 0 & 1 & 0 \\ s(\beta) & 0 & c(\beta) \end{bmatrix} \begin{bmatrix} c(\alpha) & s(\alpha) & 0 \\ -s(\alpha) & c(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

: Euler angles α, β, γ

In particular, $\hat{e}_3^{\,0}$ = z axis in inertial frame $\parallel \vec{L}$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{inert} \longrightarrow \begin{bmatrix} -c(\gamma)s(\beta) \\ s(\gamma)\,s(\beta) \\ c(\beta) \end{bmatrix}_{body} = \mathbb{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\hat{e}_3$ = symmetry axis of body

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{body} \longrightarrow \begin{bmatrix} s(\beta)c(\alpha) \\ s(\beta)\,s(\alpha) \\ c(\beta) \end{bmatrix}_{inert} = \mathbb{R}^{+}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Think I just got these mixed up at one point

Do get $T = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})^2$

for $I_1 = I_2$

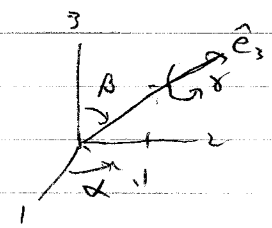
Last time, found solution for motion

$$\alpha(t) = \alpha(0) + \frac{I_3}{I_1 \cos\beta}\,\omega_3$$

$$\beta(t) = \beta(0)$$

$$\gamma(t) = \gamma(0) + \frac{I_1 - I_3}{I_1}\,\omega_3$$

$\hat{e}_3$ precesses around $\vec{L}$, described by α



rate $\dot{\alpha} = \dfrac{I_3 \omega_3}{I_1 \cos\beta} = \dfrac{L_3}{I_1 \cos\beta}$

$$= \frac{|\vec{L}| \cos\beta}{I_1 \cos\beta}$$

$$= \frac{|\vec{L}|}{I_1}$$

Compare to body frame result:

$\vec{L}$ precesses around $\hat{e}_3$ at rate $\Omega = \dfrac{I_3 - I_1}{I_1}\,\omega_3$

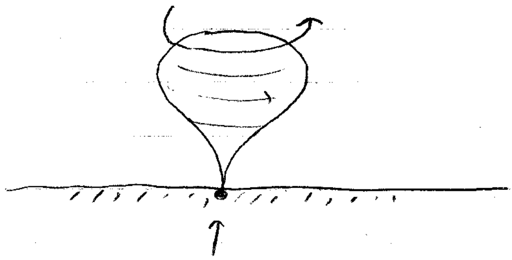
see $\Omega = -\dot{\gamma}$

Makes sense, that's what $\vec{L}$ looks like from body frame

$\rightarrow$ solution in inertial frame clearer

Note that <u>total</u> spin rate around $\hat{e}_3$ is $\omega_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$

Today move on to top in gravity
like the toy



fixed point

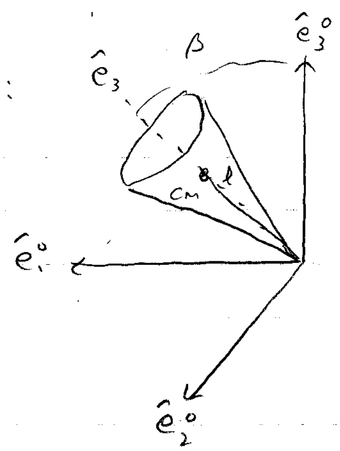
To treat, first decide where to place origin
Two choices:

a) center of mass
b) point fixed in inertial frame

If we use (a), need to deal w/ reaction forces keeping tip fixed

Better to use (b)

Set up:



l = distance from CM to tip

Get potential energy
$U = Mgl\cos\beta$

mass M

Lagrangian $L = T - U$

$$L = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})^2 - Mgl\cos\beta$$

See that α, γ still don't appear
$\Rightarrow$ p_α, p_γ = constants

Recall $p_\alpha = \vec{L} \cdot \hat{e}_3^\circ$ $\qquad p_\gamma = \vec{L} \cdot \hat{e}_3$

Here we made $\hat{e}_3^\circ$ vertical, not necessarily along $\vec{L}$

So don't have $p_\alpha = |\vec{L}|$

and $|\vec{L}| \neq$ constant

Equations of motion

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_3 (\dot{\alpha} \cos\beta + \dot{\gamma}) = \text{const}$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_1 \dot{\alpha} \sin^2\beta + p_\gamma \cos\beta = \text{const}$$

$$I_1 \ddot{\beta} = I_1 \dot{\alpha}^2 \sin\beta\cos\beta - I_3 (\dot{\alpha} \cos\beta + \dot{\gamma}) \dot{\alpha} \sin\beta + Mgl \sin\beta$$

No longer have $\beta =$ const, in general

Try to solve:

First, eliminate $\dot{\alpha}$ and $\dot{\gamma}$ using p_α, p_γ

Get
$$\dot{\alpha} = \frac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}$$

$$\dot{\gamma} = \frac{p_\gamma}{I_3} - \frac{\cos\beta (p_\alpha - p_\gamma \cos\beta)}{I_1 \sin^2\beta}$$

Plug into eqn for β ... do some algebra

$$\boxed{I_1 \ddot{\beta} = \frac{\cos\beta}{I_1 \sin^3\beta} \left(p_\alpha^2 - 2 p_\alpha p_\gamma \cos\beta + p_\gamma^2 \right) - \frac{p_\alpha p_\gamma}{I_1 \sin\beta} + Mgl \sin\beta}$$

Hopeless to solve directly

But, notice form $I_1 \ddot{\beta} = f(\beta)$

Looks like generic 1D problem
$f(\beta) =$ generalized force
(here torque)

Can solve just like central force problem in Ch 1

Know $f(\beta)$ is conservative, since

$$f(\beta) = \frac{d}{d\beta} \int f(\beta)\, d\beta$$

So β acts like motion of particle in potential

$$V_{eff}(\beta) = -\int f(\beta)\, d\beta$$

For central force problem, used

$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

to get $t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - V_{eff}}}$

To apply here, need to get V_{eff}

Integrating $f(\beta)$ looks hard

Instead, calculate Hamiltonian

Expect $H = \frac{1}{2} I_1 \dot{\beta}^2 + V_{eff}(\beta)$

In general $\quad H = \sum_{\alpha\beta\gamma} p_\sigma \dot{q}_\sigma - L$

But, if no time dependent constraints, know also

$$H = T + V$$

So we get directly

$$H = \frac{1}{2} I_1 (\dot{\alpha} \sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos\beta + \dot{\gamma})^2$$

$$+ Mgl \cos\beta$$

Want to express as function of β only
so substitute p_α, p_γ for $\dot{\alpha}, \dot{\gamma}$

Get $\boxed{H = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2 I_1 \sin^2\beta} + \frac{p_\gamma^2}{2 I_3} + Mgl\cos\beta}$

$$= \frac{1}{2} I_1 \dot{\beta}^2 + V_{eff}(\beta)$$

$$= E \quad \text{constant}$$

Formally, have solution $\quad t = \sqrt{\frac{I_1}{2}} \int \frac{d\beta}{\sqrt{E - V_{eff}(\beta)}}$

but can't do integral

Informally, knowing V_{eff} gives pretty good intuition about what β does

But first warning:

Don't try to get V_{eff} by looking at $L = T - V$

Gives wrong sign

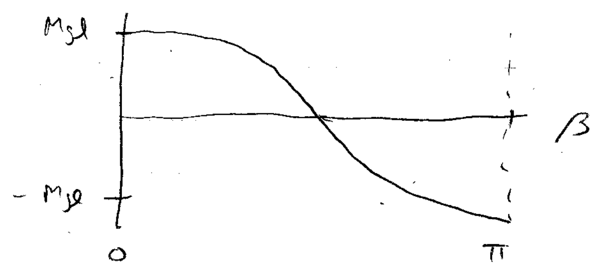
Need to use H

Ch 6 deals more directly with this issue

Look at $V_{eff}(\beta)$. Three parts:

a) $\frac{p_\gamma^2}{2I_3}$ = constant, ignore

b) $Mgl \cos\beta$



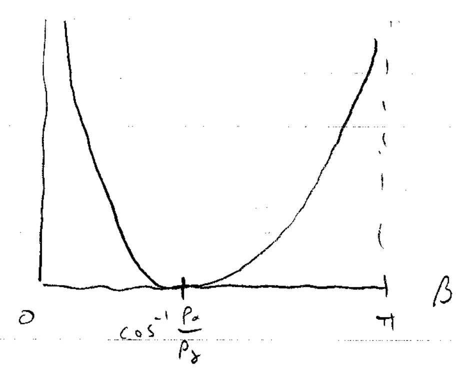
c) $\frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta}$ $\to \infty$ as $\beta \to 0, \pi$

(like angular momentum barrier)

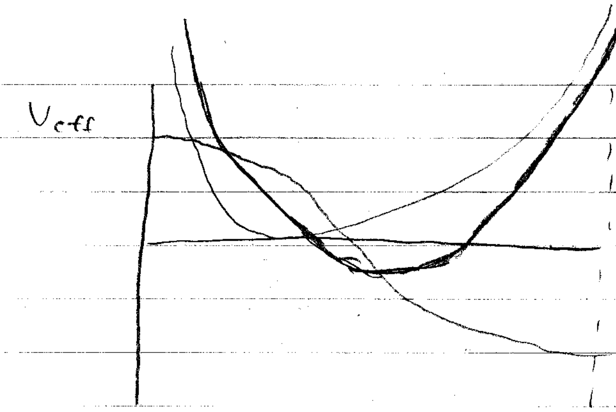
$\to 0$ at $\cos\beta = \frac{p_\alpha}{p_\gamma}$

if $p_\gamma > p_\alpha$ = case of rapidly spinning top

So:



Add (b) & (c):



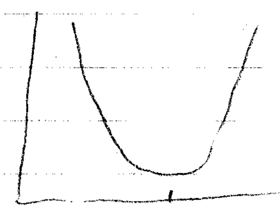
Relatively simple:

- One minimum
- For any E, constrained by β_{min} and β_{max}
- ∴ Expect periodic oscillation.

Also get solutions $\beta = 0$ (top straight up) when $p_\alpha = p_\gamma$

Explore this in HW

Look at $p_\alpha < p_\gamma$ now:



Stable precession at $\beta = \beta_0$

Solution to $\frac{\partial V_{eff}}{\partial \beta} = 0$

But we know $\frac{\partial V_{eff}}{\partial \beta} = -f(\beta)$

So have
$$f(\beta_0) = 0 = \frac{\cos\beta_0}{I_1 \sin^3\beta_0}\left(p_\alpha^2 - 2p_\alpha p_\gamma \cos\beta_0 + p_\gamma^2\right) - \frac{p_\alpha p_\gamma}{I_1 \sin\beta_0} + Mgl \sin\beta_0$$

$\beta_0 \neq 0$:
$$\boxed{\cos\beta_0\left(p_\alpha^2 - 2p_\alpha p_\gamma \cos\beta_0 + p_\gamma^2\right) = p_\alpha p_\gamma \sin^2\beta_0 - MglI_1 \sin^4\beta_0}$$

Can't solve explicitly.
But can look at small oscillations around equilibrium

Say $\beta = \beta_0 + \zeta$

$$H \approx \tfrac{1}{2} I_1 \dot{\zeta}^2 + V_{eff}(\beta_0) + \tfrac{1}{2}\zeta^2 \left.\frac{\partial^2 V_{eff}}{\partial \beta^2}\right|_{\beta_0}$$

by Taylor expansion

Compare to harmonic oscillator, see freq

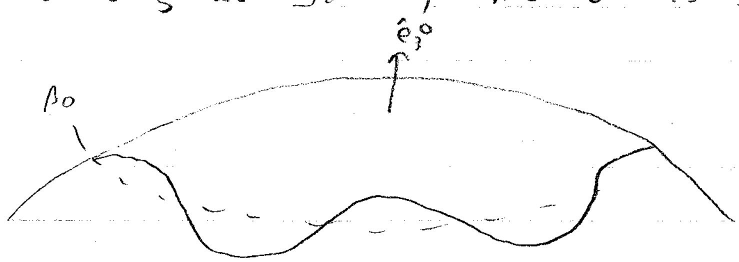
$$\Omega^2 = \frac{1}{I_1}\frac{\partial^2 V}{\partial \beta^2}$$

Can evaluate derivative

$$\left.\frac{\partial^2 V_{eff}}{\partial \beta^2}\right|_{\beta_0} = -Mgl\cos\beta_0 - \frac{3 p_\alpha p_\gamma \cos\beta_0}{I_1 \sin^2\beta_0} + \underbrace{\left(p_\alpha^2 - 2p_\alpha p_\gamma \cos\beta_0 + p_\gamma^2\right)}_{\substack{= p_\alpha p_\gamma \sin^2\beta_0 \\ - MglI_1 \sin^4\beta_0}} \frac{3 - 2\sin^2\beta_0}{I_1 \sin^4\beta_0}$$

Gives $$\boxed{\Omega^2 = \frac{p_\alpha p_\gamma - MglI_1(4 - 3\sin^2\beta_0)}{I_1 \cos^2\beta_0}}$$

As long as $\Omega^2 > 0$, motion is stable



"nutation"

If we won't, can work out $\dot{\alpha}$ and $\dot{\gamma}$ from p_γ, p_γ

In steady state, $\dot{\alpha} = \dfrac{p_\alpha - p_\gamma \cos\beta_0}{I_1 \sin^2\beta_0}$

$= $ precession rate

Can see all sorts of interesting behavior in demo!