Lecture 17

Still thinking about rigid body rotation

Specifically, <u>torque-free top</u>

Last time, considered from body frame

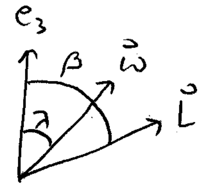
Euler's eqns $\rightarrow$ $\vec{\omega}(t)$, $\vec{L}(t)$ in body frame

Related back to inertial frame using $\vec{L}$ = const

Recall results:

$\vec{L}$ & $\vec{\omega}$ precess around $\hat{e}_3$ = body symmetry axis

rate $\Omega = \omega_3 \dfrac{I_3 - I_1}{I_1}$



$I_3 \tan\beta = I_1 \tan\lambda$

Today, revisit problem from inertial frame

Method of <u>Euler angles</u>

More straightforward, but harder math:

1) Set up coords describing orientation of object
2) Calculate Lagrangian
3) Get eqns of motion

I.) Set up coords

Have 3 degrees of freedom for rotation

Tempting to say coords = $\Theta_1, \Theta_2, \Theta_3$

Θ_i = rotation angle about axis i

Then have $\vec{\omega} = \dot{\vec{\Theta}}$, sounds good...

But doesn't work

Problem is that order of rotations matters

Rotating Θ_1 about $\hat{e}_1$ then Θ_2 about $\hat{e}_2$

$\neq$ Θ_2 about $\hat{e}_2$ then Θ_1 about $\hat{e}_1$

Demo...

Say that rotations don't commute

Could add instruction to do in order $\Theta_1, \Theta_2, \Theta_3$ ... OK

Also need to be clear about whether Θ_i corresponds to rotation about body axis $\hat{e}_i$ or inertial axis $\hat{e}_i^o$

Could specify inertial axes $\hat{e}_i^o$ ... OK

But if you use these coords, Lagrangian comes out very messy

Euler figured out coords that make L as simple as possible

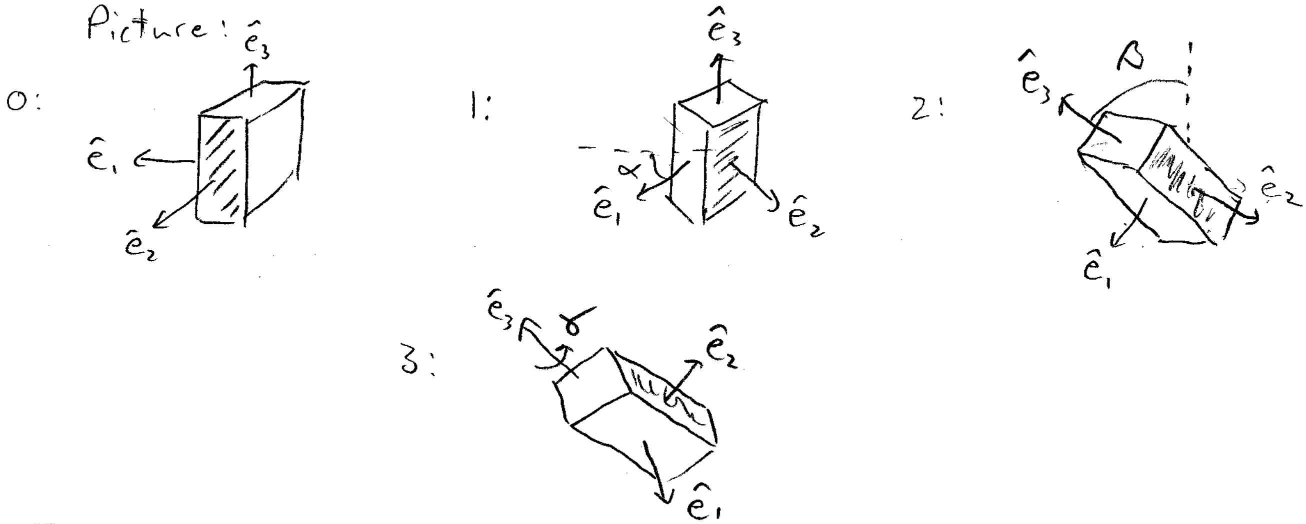
Use them instead!

Euler angles α, β, γ

Instructions

0. Start with body axes aligned to inertial axes
1. Rotate about $\hat{e}_3^o = \hat{e}_3$ by α
2. Rotate about $\hat{e}_2$ by β
3. Rotate about $\hat{e}_3$ by γ

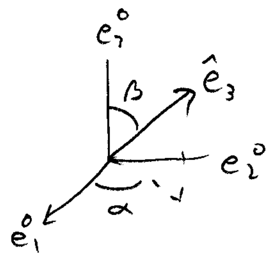
Picture:



Some notes:

- Easy way to think about

  (α, β) are regular spherical angles (ϕ, θ)
  for $\hat{e}_3$ w/ respect to inertial frame



Then γ = rotation about $\hat{e}_3$

- Terminology: "line of nodes"
  = position of $\hat{e}_2$ axis after step 1
  = axis of rotation for β

- Even though instructions give sequence,
  angles are independent coords

To use Euler angles, useful to have rotation matrix
for inertial → body frame

Recall

$$\mathbb{R}_{body \leftarrow inert} = \begin{bmatrix} \hat{e}_1 \cdot \hat{e}_1^o & \hat{e}_1 \cdot \hat{e}_2^o & \hat{e}_1 \cdot \hat{e}_3^o \\ \hat{e}_2 \cdot \hat{e}_1^o & \hat{e}_2 \cdot \hat{e}_2^o & \hat{e}_2 \cdot \hat{e}_3^o \\ \hat{e}_3 \cdot \hat{e}_1^o & \hat{e}_3 \cdot \hat{e}_2^o & \hat{e}_3 \cdot \hat{e}_3^o \end{bmatrix}$$

gives $\vec{A}_{body} = \mathbb{R} \vec{A}_{inert}$

Not easy to evaluate directly
Easier to compose from individual rotations

$$\mathbb{R} = \underbrace{\begin{bmatrix} c(\gamma) & s(\gamma) & 0 \\ -s(\gamma) & c(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Rotate about } \hat{e}_3 \text{ by } \gamma \\ \mathbb{R}_3(\gamma)}} \underbrace{\begin{bmatrix} c(\beta) & 0 & -s(\beta) \\ 0 & 1 & 0 \\ s(\beta) & 0 & c(\beta) \end{bmatrix}}_{\substack{\text{Rotate about } \hat{e}_2 \text{ by } \beta \\ \mathbb{R}_2(\beta)}} \underbrace{\begin{bmatrix} c(\alpha) & s(\alpha) & 0 \\ -s(\alpha) & c(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Rotate about } \hat{e}_3 \text{ by } \alpha \\ \mathbb{R}_1(\alpha)}}$$

Multiplying out gives big mess. Easier to leave as product $\mathbb{R}_3 \mathbb{R}_2 \mathbb{R}_1$

For instance, say $\vec{L} = L\hat{e}_3^{\circ}$ in inertial frame

In body frame,

$$(\vec{L})_{body} = \mathbb{R}\,(\vec{L})_{inert}$$

$$\mathbb{R}_1 \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix}$$

$$\mathbb{R}_2 \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} = L \begin{bmatrix} -\sin\beta \\ 0 \\ \cos\beta \end{bmatrix}$$

$$\mathbb{R}_3 L \begin{bmatrix} -\sin\beta \\ 0 \\ \cos\beta \end{bmatrix} = L \begin{bmatrix} -\cos\gamma\sin\beta \\ \sin\gamma\sin\beta \\ \cos\beta \end{bmatrix} = \vec{L}$$

More often, need to go from body frame to inertial

$$\mathbb{R}_{inert \leftarrow body} = (\mathbb{R}_{body \leftarrow inert})^{+}$$

$$= \mathbb{R}_1^{+}(\alpha)\,\mathbb{R}_2^{+}(\beta)\,\mathbb{R}_3^{+}(\gamma)$$

Use this, for instance, to get body axes in inertial frame

$$(\hat{e}_1)_{inert} = \mathbb{R}^{+}(\hat{e}_1)_{body} = \mathbb{R}^{+}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha \\ \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha \\ -\cos\gamma\sin\beta \end{bmatrix}$$

$$(\hat{e}_2)_{inert} = \begin{bmatrix} -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha \\ -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha \\ \sin\gamma\sin\beta \end{bmatrix} \qquad (\hat{e}_3)_{inert} = \begin{bmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{bmatrix}$$

Like spherical coords, as noted!

II. Express Lagrangian in terms of (α, β, γ)

Torque-free top, $L = T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

$$= \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

when $\hat{e}_i$'s are principle axes

Need to relate ω_s's to α, β, γ

Have $\vec{\omega} = \dot{\alpha}\,\hat{e}_\alpha + \dot{\beta}\,\hat{e}_\beta + \dot{\gamma}\,\hat{e}_\gamma$

$\hat{e}_\alpha$ = rotation axis for $\alpha = \hat{e}_3^0$

$\hat{e}_\beta$ = rotation axis for β = line of nodes

$\hat{e}_\gamma$ = rotation axis for $\gamma = \hat{e}_3$

Works because α, β, γ independent

Also have $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$

Use $\mathbb{R}$ to write $\hat{e}_\alpha, \hat{e}_\beta, \hat{e}_\gamma$ in terms of $\hat{e}_1, \hat{e}_2, \hat{e}_3$

Start with $\hat{e}_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{inert}$

$$(\hat{e}_\alpha)_{body} = \mathbb{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{bmatrix}$$ did already

$$= -\sin\beta\cos\gamma\,\hat{e}_1 + \sin\beta\sin\gamma\,\hat{e}_2 + \cos\beta\,\hat{e}_3$$

Next, $\hat{e}_\beta$ = line of nodes

= "2" axis <u>after</u> first rotation

So $$\hat{e}_\beta = \mathbb{R}_3(\gamma)\,\mathbb{R}_2(\beta)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbb{R}_3(\gamma)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{bmatrix}$$

$$\hat{e}_\beta = \sin\gamma\,\hat{e}_1 + \cos\gamma\,\hat{e}_2$$

Finally, have $\hat{e}_\gamma = \hat{e}_3$ already

So,
$$\vec{\omega} = \dot{\alpha}\,\hat{e}_\alpha + \dot{\beta}\,\hat{e}_\beta + \dot{\gamma}\,\hat{e}_\gamma$$
$$= \dot{\alpha}\left(-\hat{e}_1 \sin\beta\cos\gamma + \hat{e}_2 \sin\beta\sin\gamma + \hat{e}_3 \cos\beta\right)$$
$$+ \dot{\beta}\left(\hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma\right)$$
$$+ \dot{\gamma}\,\hat{e}_3$$

$$= \hat{e}_1\left(-\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma\right) \to \omega_1$$
$$+ \hat{e}_2\left(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma\right) \to \omega_2$$
$$+ \hat{e}_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right) \to \omega_3$$

Plug in to get $T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$

$$\boxed{\begin{aligned} T = \frac{1}{2}\Big\{ & \dot{\alpha}^2 \sin^2\beta \left(I_1 \cos^2\gamma + I_2 \sin^2\gamma\right) \\ & + \dot{\beta}^2\left(I_1 \sin^2\gamma + I_2\cos^2\gamma\right) \\ & + 2\dot{\alpha}\dot{\beta}\sin\beta\sin\gamma\cos\gamma\left(I_2 - I_1\right) \\ & + \left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 I_3 \Big\}\end{aligned}}$$

Simplifies for $I_1 = I_2$:

$$\boxed{T \to \frac{1}{2} I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2}$$

Let's assume this

III Equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \qquad q = \alpha, \beta, \gamma$$

Note $\frac{\partial L}{\partial \alpha}$ and $\frac{\partial L}{\partial \gamma} = 0$

$$\Rightarrow \quad p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2\beta + I_3 \cos\beta (\dot{\alpha}\cos\beta + \dot{\gamma})$$

and $p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3(\dot{\alpha}\cos\beta + \dot{\gamma})$

are constants

See $p_\gamma = I_3 \omega_3 = L_3$: component of $\vec{L}$ along $\hat{e}_3$ = const

($\vec{L}$ precess around $\hat{e}_3$ in body frame)

also p_α = component of $\vec{L}$ along $\hat{e}_3^o$

(since $e_\alpha = \hat{e}_3^o$)

If we take $\hat{e}_3^o$ along $\vec{L}$ to begin with, then $p_\alpha = |\vec{L}|$

Remaining eqn of motion is $\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} = 0$

$$I_1 \ddot{\beta} = \dot{\alpha} \sin\beta (I_1 \dot{\alpha} \cos\beta - \underbrace{I_3 \omega_3}_{\text{constant}})$$

How to solve?

Note, if $p_\alpha = |\vec{L}|$ and $p_\gamma = L_3$

must have $p_\gamma = p_\alpha \cos\beta$

Since β = angle between $\hat{e}_3$ and $\hat{e}_3^o$

Implies $\underline{\beta = \text{constant}}$

If $\beta = \text{const}$, eqn for β becomes

$$I_1 \ddot{\beta} = 0 = \dot{\alpha} \sin\beta \left(I_1 \dot{\alpha} \cos\beta - I_3 \omega_3 \right)$$

Could have $\dot{\alpha} = 0$

Then $p_\alpha \to I_3 \dot{\gamma} \cos\beta$

$p_\gamma \to I_3 \dot{\gamma}$

But we require $p_\gamma = p_\alpha \cos\beta$

Inconsistent unless $\beta = 0$

* Could have $\beta = 0$

Body spinning around symmetry axis

Otherwise, require $\boxed{\dot{\alpha} \cos\beta = \dfrac{I_3 \omega_3}{I_1}}$

We know $\omega_3 = \dot{\alpha} \cos\beta + \dot{\gamma}$

So $\dot{\gamma} = \omega_3 - \dot{\alpha} \cos\beta = \omega_3 - \dfrac{I_3 \omega_3}{I_1}$

$$\boxed{\dot{\gamma} = \omega_3 \frac{I_1 - I_3}{I_1}}$$

Here parametrizing in terms of ω_3

= spin rate about axis

usually convenient

So full solution is

$$\alpha(t) = \alpha(0) + \frac{I_3}{I_1 \cos\beta} \omega_3$$

$$\beta(t) = \beta(0)$$

$$\gamma(t) = \gamma(0) + \frac{I_1 - I_3}{I_1} \omega_3$$

Specify orientation of body as function of time

In particular, consider $\hat{e}_3$ axis

$$\hat{e}_3 = \begin{bmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{bmatrix}$$

- Makes fixed angle β to $\vec{L} \propto \hat{e}_3^0$
- Precess around at rate $\dot{\alpha} = \dfrac{I_3\omega_3}{I_1\cos\beta}$

$$= \frac{L_3}{I_1\cos\beta} = \frac{|\vec{L}|\cos\beta}{I_1\cos\beta}$$

$$= \frac{|\vec{L}|}{I_1}$$

Compare to last lecture... $\Omega = \dfrac{I_3 - I_1}{I_1}\omega_3$

See now $\Omega = -\dot{\gamma}$

Makes sense: Ω = rate that $\vec{L}$ precessed around $\hat{e}_3$ in body frame

Since body itself spins, get component of apparent precession from that, as well as from actual motion of $\hat{e}_3$

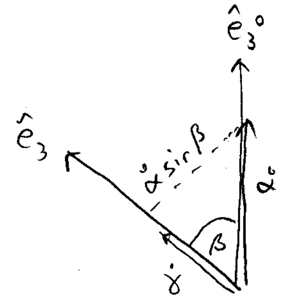
Can also get $\vec{\omega}(t)$

$$\vec{\omega} = \dot{\alpha}\hat{e}_\alpha + \dot{\gamma}\hat{e}_\gamma \qquad (\dot{\beta} = 0)$$

$$= \dot{\alpha}\underbrace{\hat{e}_3^0}_{\text{fixed}} + \dot{\gamma}\underbrace{\hat{e}_3}_{\text{precessing}}$$

So $\vec{\omega}$ also precesses around $\vec{L}$ at fixed angle

Get angle between $\vec{\omega}$ and $\vec{L}$: λ



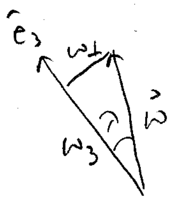
Component of $\vec{\omega}$ $\perp$ to $\hat{e}_3$

$$= \dot{\alpha}\sin\beta$$

Component of $\vec{\omega}$ $\parallel$ to $\hat{e}_3$

$$= \omega_3$$

So $$\tan\lambda = \frac{\omega_\perp}{\omega_3} = \frac{\dot{\alpha}\sin\beta}{\omega_3}$$



Use $$\dot{\alpha} = \frac{I_3\omega_3}{I_1\cos\beta}$$

$$\tan\lambda = \frac{\frac{I_3}{I_1}\omega_3\frac{\sin\beta}{\cos\beta}}{\omega_3} = \frac{I_3}{I_1}\tan\beta$$

just as before

Gives complete picture in inertial frame more easily than Euler eqns method