Lecture 16

Dynamics of rigid bodies

Last time, had rotation axis fixed

Today: start free rotation

Develop with example: "torque-free top"

- object spinning freely in space

See fairly complex behavior: demos

try to understand

Since $\vec{\Gamma} = 0$, know $\vec{L} =$ constant

(in inertial frame, w/respect to center of mass)

Relate to what body is doing

Ultimately, solve for body frame vectors

$\hat{e}_1, \hat{e}_2, \hat{e}_3\,(t)$, relative to $\vec{L}$.

We'll do that, but today take indirect approach:

Solve for $\vec{L}(t)$ relative to body frame

Considerably easier, and provides useful insight

Know how derivatives in body & inertial frames are related

$$\left(\frac{d\vec{L}}{dt}\right)_{inert} = \left(\frac{dL}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = 0$$

So $$\left(\frac{dL}{dt}\right)_{body} = -\vec{\omega} \times \vec{L}$$

Also have $\vec{L} = \mathbb{I}\,\vec{\omega}$

Use to express everything in terms of $\vec{\omega}$

Be smart: make body frame = principle axes

moments I_1, I_2, I_3

Then $L_s = I_s \omega_s \qquad s = 1, 2, 3$

$$\frac{dL_s}{dt} = I_s \frac{d\omega_s}{dt} = -(\vec{\omega} \times \vec{L}) \cdot \hat{e}_s$$

$s = 1$:

$$I_1 \dot{\omega}_1 = -(\vec{\omega} \times \vec{L}) \cdot \hat{e}_1$$

$$= -(\omega_2 L_3 - \omega_3 L_2)$$

$$= -(\omega_2 \omega_3 I_3 - \omega_3 \omega_2 I_2)$$

$$\boxed{I_1 \dot{\omega}_1 = \omega_2 \omega_2 (I_2 - I_3)}$$

↳ derivitave in body frame!

Similarly, get

$$\boxed{\begin{aligned} I_2 \dot{\omega}_2 &= \omega_1 \omega_3 (I_3 - I_1) \\ I_3 \dot{\omega}_3 &= \omega_1 \omega_2 (I_1 - I_2) \end{aligned}}$$

Called Euler's Equations

See that dynamics depends a lot on symmetry of body

"Sphere": $I_1 = I_2 = I_3 = I$

Then $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0 \Rightarrow \vec{\omega} = \text{constant}$

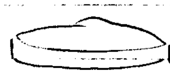
and $\vec{L} = I\vec{\omega} = \text{constant}$

Know $\vec{L} = \text{const}$ in inertial frame
have $\vec{L} = \text{const}$ in body frame as well

$\Rightarrow$ no precession

"Symmetric Top": $I_1 = I_2 \neq I_3$

football  $I_3 < I_1$

or frisbee  $I_3 > I_1$

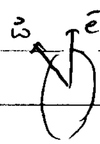
Euler equations

$$\dot{\omega}_3 = 0$$

$$\dot{\omega}_1 = -\omega_2\omega_3 \frac{I_3 - I_1}{I_1}$$

$$\dot{\omega}_2 = +\omega_1\omega_3 \frac{I_3 - I_1}{I_2}$$

So $\omega_3 = \text{constant}$
= component of $\vec{\omega}$ along $\hat{e}_3$ axis



= rate of spin about symmetry axis

Also have $\Omega \equiv \frac{I_3 - I_1}{I_1} \omega_3$ = constant

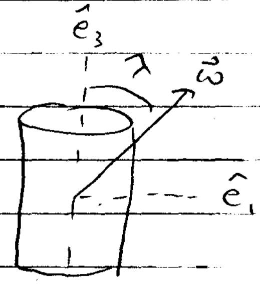
and $\dot{\omega}_1 = -\Omega \omega_2$
$\dot{\omega}_2 = \Omega \omega_1$

$\ddot{\omega}_1 = -\Omega \dot{\omega}_2 = -\Omega^2 \omega_1$

$\omega_1(t)$: harmonic oscillation at freq Ω

Same for ω_2.

Suppose initial conditions
$\omega_1(0) = \omega \sin\lambda$
$\omega_2(0) = 0$
$\omega_3(0) = \omega \cos\lambda$



Then $\omega_1(t) = \omega \sin\lambda \cos\Omega t$
$\omega_2(t) = \omega \sin\lambda \sin\Omega t$
$\omega_3(t) = \omega \cos\lambda$

$\vec{\omega}$ precesses around $\hat{e}_3$ at rate Ω
That is precession we observe

That's in body frame. relate to inertial frame?
Know $\vec{\omega}$ precessing around $\hat{e}_3$
But $\hat{e}_3$ precessing around fixed $\vec{L}$

Let's get $\vec{L}$ in body frame

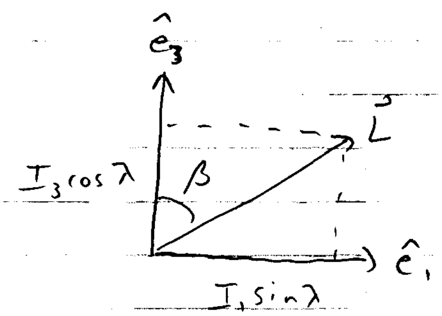
$$\vec{L} = \mathbb{I}\vec{\omega} = \begin{bmatrix} I_1 & & 0 \\ & I_1 & \\ 0 & & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$(\vec{L})_{body} = \begin{bmatrix} I_1 \omega \sin\lambda \cos\omega t \\ I_1 \omega \sin\lambda \sin\omega t \\ I_3 \omega \cos\lambda \end{bmatrix}$$

Also precesses around $\hat{e}_3$

But at different angle than $\vec{\omega}$

$t=0$:

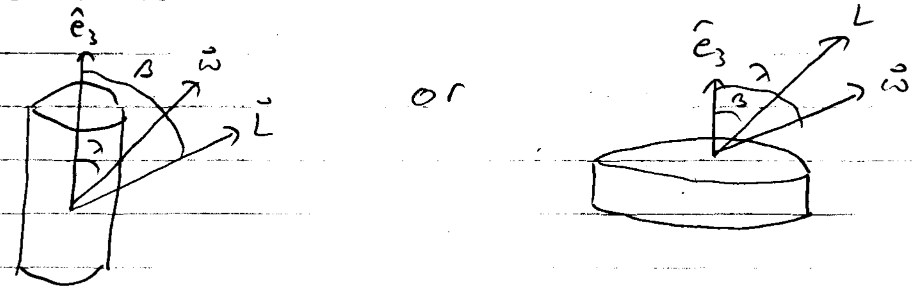


$$\tan\beta = \frac{I_1 \sin\lambda}{I_3 \cos\lambda} = \frac{I_1}{I_3} \tan\lambda$$

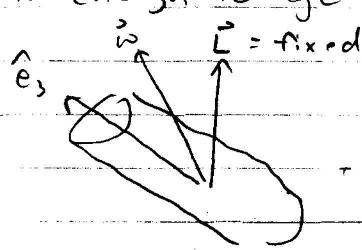
If $I_1 < I_3$ (football) $\Rightarrow$ $\beta > \lambda$

$I_1 > I_3$ (frisbee) $\Rightarrow$ $\beta < \lambda$

So picture is



That is enough to get inertial frame picture:



$\hat{e}_3$ makes constant angle β to $\vec{L}$

precesses around at rate

$$\Omega = \frac{I_3 - I_1}{I_1} \omega_3 = \frac{I_3 - I_1}{I_1} \omega \cos\lambda$$

Still wonder why?

First, terminology:

"Precession" = steady rotation of one vector around another

Here refers to precession of $\hat{e}_3$ around $\vec{L}$

(Not like for tops in gravity: precession of $\vec{L}$ around $\vec{g}$)

Here precession not driven

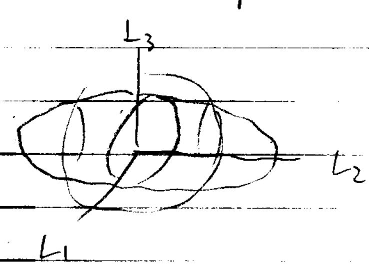
Get some insight from energy

Have $E = T = \frac{1}{2} \sum_s I_s \omega_s^2$

$$= \frac{1}{2}\left[I_1(\omega_1^2+\omega_2^2)+I_3\omega_3^2\right]$$

$$= \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_1} + \frac{L_3^2}{2I_3} = \text{constant}$$

Defines ellipsoid in "$\vec{L}$-space"



Also have $L^2 = L_1^2+L_2^2+L_3^2$ = const

= sphere in L space

Intersection of sphere & ellipsoid usually two circles

$\vec{L}$ confined to circles

So only possible motion of $\vec{L}$ (in body frame)

= precession around $\hat{e}_3$

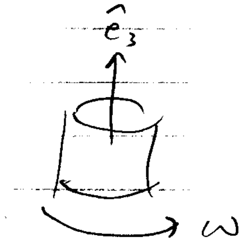
OK, but why precess?

T • L^2 conserved if $\vec{L}$ sat still

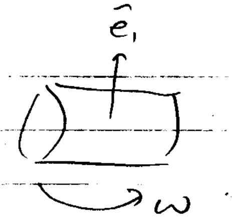
Think about what $\vec{\omega}$ is.

ω_s = angular velocity about axis $\hat{e}_s$

If $\omega_1 = \omega_2 = 0$, $\omega_3 \neq 0$:

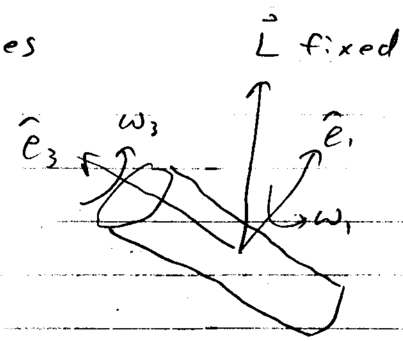


If $\omega_2 = \omega_3 = 0$, $\omega_1 \neq 0$:



If $\omega_2 = 0$, $\omega_1, \omega_3 \neq 0$

have rotation about both axes



If body weren't precessing, wouldn't have ω_1 component

- Really just kinematics!

  precession is what ω_1 & ω_3 together describe

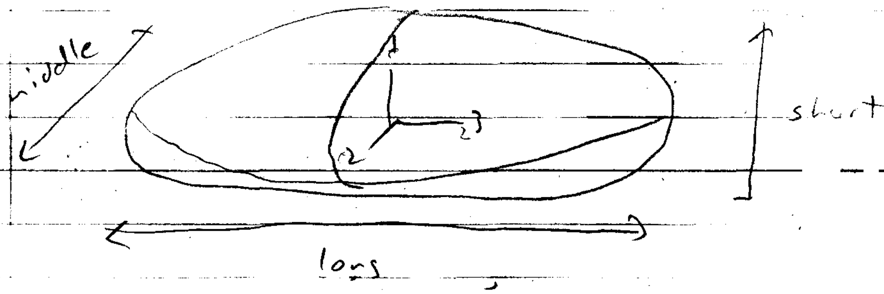
Say something about asymmetric top: $I_1 < I_2 < I_3$

Motion is complicated, can't solve analytically

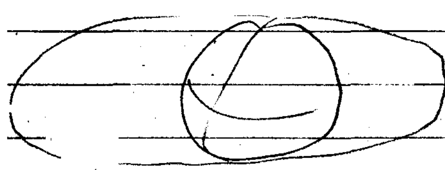
Get insight from $T - L^2$ construction

Still have sphere $L^2 =$ const

Now $T =$ const defines asymmetric ellipsoid

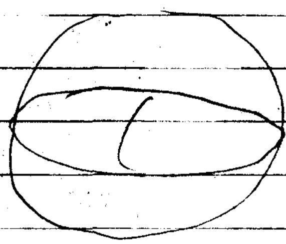


If $\vec{L} \approx L\hat{e}_1$, lowest possible T for given L:



Motion stable

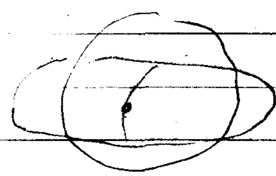
If $\vec{L} \sim L\hat{e}_3$, max possible T



Motion stable

If $\vec{L} \sim L\hat{e}_2$, intersection complicated

Motion unstable



Demo

Can solve analytically for $\vec{L}$ near axis

$$\text{Say} \quad \vec{L} \approx L_0 \hat{e}_3 \Rightarrow \omega \approx \omega_0 \hat{e}_3$$

Treat like small oscillations

$$\omega_1 = \zeta_1(t)$$
$$\omega_2 = \zeta_2(t)$$
$$\omega_3 = \omega_0 + \zeta_3(t)$$

Linearize eqns for small ζ

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$$

$$I_1 \dot{\zeta}_1 = \zeta_2 (\omega_0 + \zeta_3)(I_2 - I_3)$$

$$\approx \zeta_2 \omega_0 (I_2 - I_3)$$

Similarly $\quad I_2 \dot{\zeta}_2 \approx \zeta_1 \omega_0 (I_3 - I_1)$

but $\quad I_3 \dot{\zeta}_3 = \zeta_1 \zeta_2 (I_1 - I_2) \approx 0$

So $\omega_3 \approx$ constant

Combine eqns for ζ_1 and ζ_2:

$$\ddot{\zeta}_1 = \dot{\zeta}_2 \omega_0 \frac{I_2 - I_3}{I_1}$$

$$= \zeta_1 \omega_0^2 \frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_2}$$

$$\ddot{\zeta}_1 = -\Omega_3^2 \zeta_1 \qquad \text{for} \quad \boxed{\Omega_3^2 = \omega_0^2 \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}}$$

$\Omega_3^2 > 0$ ✓

Similarly, $\ddot{\zeta}_2 = -\Omega_3^2 \zeta_2$

So here $\vec{\omega}$ precesses around $\hat{e}_3$, as expected

Can look at rotations around other axes
by cycling through 1, 2, 3

If $\vec{\omega} \approx \omega_0 \hat{e}_1$, get precession at

~~3→1~~
2→3
1→2

$$\Omega_1^2 = \omega_0^2 \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}$$

> 0, stable precession

If $\vec{\omega} \approx \omega_0 \hat{e}_2$ get

3→2
2→1
1→3

$$\Omega_2^2 = \omega_0^2 \frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3} < 0$$

motion is not stable

More general motion requires numerical integration

But not very easy to relate back to inertial frame

Next time, develop method to get motion
relative to inertial frame directly