Lecture 15

Midterm: Average 21/40
High = 36 Low = 7

A little disappointed about #2...

Also have estimated grade to date, including HW

Back to Ch 5

Discussing inertia tensor I_{ij} for a rigid body:

$$I_{ij} = \int \rho (r^2 \delta_{ij} - r_i r_j)\, d^3r$$

$\rho(\vec{r})$ = mass density

Diagonal terms $I_{ii} = \int \rho (r_{i-1}^2 + r_{i+1}^2)\, d^3r$

Off diagonal $I_{ij} = \int \rho\, r_i r_j\, d^3r$

Gives
$$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$$
$$L_i = \sum_j I_{ij} \omega_j$$

in terms of angular velocity of body $\vec{\omega}$

Usually best to calculate $\bar{I}_{ij}$ = inertia tensor with respect to center of mass

Then relative to center at $\vec{a}$, get

$$\boxed{I_{ij} = \bar{I}_{ij} + M(\delta_{ij} a^2 - a_i a_j)}$$

That lets us translate coords... what about rotating them?

Transforms just like any other matrix does

Important to understand: go through

First, consider vector $\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$

rotated coords $\rightarrow$ $a_1'\hat{e}_1' + a_2'\hat{e}_2' + a_3'\hat{e}_3'$

Given (a_1, a_2, a_3), what are (a_1', a_2', a_3')?

Know $a_i' = \hat{e}_i' \cdot \vec{a} = \sum_j \hat{e}_i' \cdot \hat{e}_j \, a_j$

$$= \sum_j R_{ij} \, a_j$$

$$R_{ij} = \hat{e}_i' \cdot \hat{e}_j$$

"rotation matrix"

or, $\vec{a}' = \mathbb{R}\vec{a}$

If $\mathbb{R}$ describes a rotation, then it is orthonormal:

$$\mathbb{R}^{+} = \mathbb{R}^{-1}$$

Because $\vec{a}' \cdot \vec{a}' = (\vec{a})^{+}(\vec{a}) = \vec{a}^{+}\mathbb{R}^{+}\mathbb{R}\vec{a} = \vec{a}^{+}\vec{a}$ for any $\vec{a}$

Requires $\mathbb{R}^{+}\mathbb{R} = \mathbb{1}$

For a matrix, consider arb relation $\vec{b} = \mathbb{M}\vec{a}$

Rotate coords, need $\vec{b}' = \mathbb{M}'\vec{a}'$

$$\mathbb{R}\vec{b} = \mathbb{M}'\mathbb{R}\vec{a}$$

$$\vec{b} = \mathbb{R}^{+}\mathbb{M}'\mathbb{R}\vec{a}$$

so $\mathbb{M} = \mathbb{R}^{+}\mathbb{M}'\mathbb{R}$

or $\boxed{\mathbb{M}' = \mathbb{R}\mathbb{M}\mathbb{R}^{+}}$

So if you calculate I_{ij} in one frame, need in rotated frame, use

$$I'_{ij} = \sum_{kl} R_{ik} I_{kl} R_{jl}$$

again, $R_{ij} = \hat{e}'_i \cdot \hat{e}_j$

So, if we calculate I_{ij} in one frame, can easy get in any frame

Turns out, there is a preferred frame where I_{ij} is simplest

$\equiv$ principle axes

Start by noting $I_{ij} = I_{ji}$ ... symmetric matrix

Any real symmetric matrix can be diagonalized

$$\det(\mathbb{I} - \lambda \mathbb{1}) = 0$$

Roots $\lambda = I_1, I_2, I_3$: principle moments of inertia

Corresponding eigenvectors = principle axes

If we align coord frame to axes, $\mathbb{I}$ is diagonal

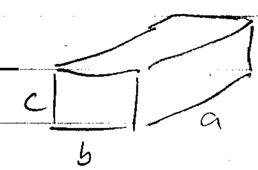
$$\overline{\overline{\mathbb{I}}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

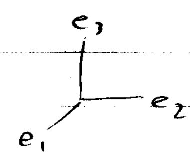
Principle axes correspond to symmetry axes

- good to set up coords along symmetry to start

Common building blocks:

Rectangular block





$$\overline{\overline{\mathbb{I}}} = \frac{M}{12} \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

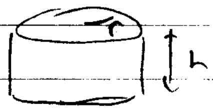
$$\overline{\overline{\mathbb{I}}}_{cube} = \frac{M}{6} \begin{bmatrix} a^2 & & 0 \\ & a^2 & \\ 0 & & a^2 \end{bmatrix} = \text{isotropic}$$

<u>any</u> axis equivalent

Sphere, radius a

$$\mathbb{I}_{sphere} = \frac{2}{5} M a^2 \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

Cylinder



$$\overline{\mathbb{I}} = \frac{M}{12} \begin{bmatrix} 3r^2+h^2 & 0 & 0 \\ 0 & 3r^2+h^2 & 0 \\ 0 & 0 & 6r^2 \end{bmatrix}$$

Hoop, radius r:

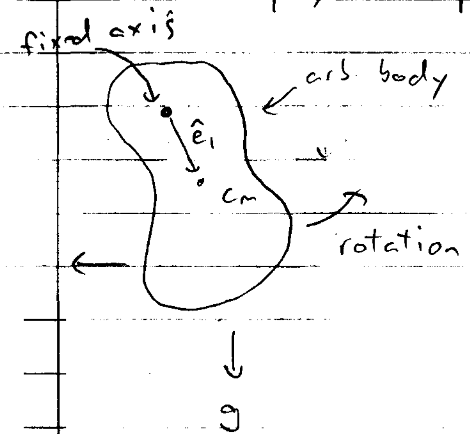
$$\mathbb{I} = M \begin{bmatrix} \frac{1}{2}r^2 & 0 & 0 \\ 0 & \frac{1}{2}r^2 & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

Can use these to construct more complex objects using composition, translation, + rotation properties

So, we can hopefully calculate $\mathbb{I}$.
What can we do with it?
→ solve for dynamics of body

· Generally a 3D problem... pretty hard

Start easy: constrain motion to one axis: "physical pendulum" swings in gravity



Analyze using

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \text{external torque}$$

Derived in first lecture, for arbitrary system

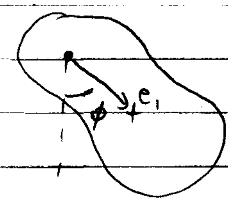
But requires either
1) origin fixed in inertial frame
or 2) origin = center of mass

Pendulum problem is type (1)... put origin at axis

Set up body coords: $\hat{e}_3$ along rotation axis
$\hat{e}_1$ points from axis to CM
$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1$

$\hat{e}_3$ is fixed $= \hat{e}_3^{\,0}$

Then $\left(\frac{dL_3}{dt}\right)_{inert} = \left(\frac{dL_3}{dt}\right)_{body} = \Gamma_3$



$$L_3 = \sum_j I_{3j}\,\omega_j$$

$\hookrightarrow$ generally not diagonal

But here $\omega_1 = \omega_2 = 0$

can only rotate around $\hat{e}_3$

So $L_3 = I_{33}\,\omega_3$

$\omega_3 = \dot{\phi}$

$$\boxed{\text{Important: } I_{jj} \text{ gives moment of inertia for rotation about axis } \hat{e}_j}$$

Here $I_{33}\,\ddot{\phi} = \Gamma_3$

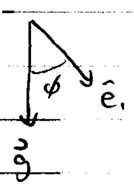
$\vec{\Gamma}$ = from gravity

$$= \int \rho(\vec{r})\,\vec{r}\times\vec{g}\,d^3r = M\vec{R}\times\vec{g}$$

$\vec{R}$ = center of mass

If COM is distance l from axis, $\vec{R} = l\hat{e}_1$

$$\vec{\Gamma} = Ml\,\hat{e}_1\times\vec{g} = Mlg\sin\phi\ \hat{\otimes}$$



$$= -Mlg\sin\phi\ \hat{e}_3$$

so $\Gamma_3 = -Mlg\sin\phi$

So $\quad I_{33}\ddot{\phi} = -Mlg\sin\phi$

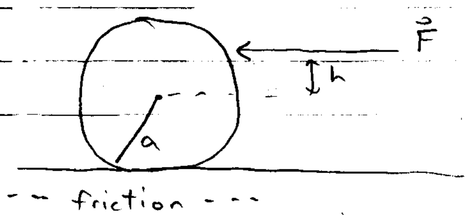
$$\boxed{\ddot{\phi} + \frac{Mlg}{I_{33}}\sin\phi = 0}$$

Pendulum equation, frequency $\Omega^2 = \dfrac{Mlg}{I_{33}}$

For particle at COM, $I_{33} = Ml^2$

$\Omega^2 \to \dfrac{g}{l}$, as expected

Another example: Billiard ball

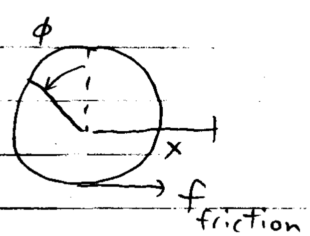


Apply force $\vec{F}$, distance h above center

Will ball roll or slide?

(assume $\vec{F}$ centered, so no side spin: rotation axis fixed)

Now two coords $\quad x =$ position of CM
$\phi =$ rotation angle



Friction force $= \mu Mg$

Newton gives

$$\boxed{M\ddot{x} = -\mu Mg + F}$$

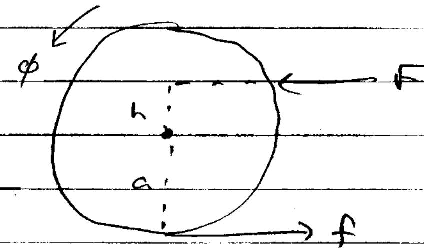
For ϕ, use $\dfrac{d\vec{L}}{dt} = \vec{\Gamma}$

Here need to put origin at center of mass, since body is accelerating

Then $L = I\omega$ $\qquad I = \frac{2}{5} M a^2$

$\omega = \dot{\phi}$

Get torques from diagram



For $h > 0$, both torques > 0

$$\Gamma = hF + af$$

So $\boxed{I\ddot{\phi} = hF + \mu M g a}$

Solve for impulsive force $F(t)$, acts for very short time

Define impulse

$$J = \int F(t)\,dt$$

Ball initially at rest

After impulse, have $p_x = J$

$\rightarrow$ So center of mass moves with $v_0 = \frac{J}{M}$

Also imparts angular momentum $L = hJ$

$\rightarrow$ angular velocity $\omega_0 = \frac{hJ}{I}$

Use v_0 & ω_0 as initial condition for free motion

$\ddot{x} = -\mu g$ $\qquad \ddot{\phi} = \frac{\mu M g a}{I} = \frac{5}{2}\frac{\mu g}{a}$

Solutions $\quad x(t) = v_0 t - \frac{1}{2}\mu g t^2$

$$\phi(t) = \omega_0 t + \frac{1}{2}\frac{\mu M g a}{I}t^2$$

This assumes ball is sliding

Eventually, reach rolling condition $\dot{x} = a\dot{\phi}$

Then ball starts rolling and friction stops

Solve for time t_1 when this occurs:

Have
$$\dot{x} = v_0 - \mu g t$$
$$a\dot{\phi} = a\omega_0 + \frac{\mu M g a^2}{I}t$$

Equate:
$$v_0 - \mu g t_1 = a\omega_0 - \frac{\mu M g a^2}{I}t_1$$

$$v_0 - a\omega_0 = \mu g\left(1 + \frac{Ma^2}{I}\right)t_1$$

Have $\quad v_0 = \frac{J}{M} \quad , \omega_0 = \frac{hJ}{I} = v_0 \frac{Mh}{I}$

$$\frac{J}{M}\left(1 - \frac{Mha}{I}\right) = \mu g\left(1 + \frac{Ma^2}{I}\right)t_1$$

$$\boxed{t_1 = \frac{v_0}{\mu g}\left(\frac{I - Mha}{I + Ma^2}\right)}$$

Sphere $I = \frac{2}{5}Ma^2 \qquad t_1 = \frac{v_0}{\mu g}\left(\frac{\frac{2}{5}a - h}{\frac{2}{5}a + a}\right)$

$$\left(\frac{5}{7}\right)\left(\frac{2}{5} - \frac{h}{a}\right)$$

$$t_1 = \frac{2}{7}\frac{v_0}{\mu g}\left(1 - \frac{5h}{2a}\right)$$

If $h < \frac{2}{5}a$, $t_1 > 0$ ball slides for a bit, then rolls

If $h = \frac{2}{5}a$ $t_1 = 0$ ball rolls immediately

If $h > \frac{2}{5}a$, $t_1 < 0$?

Here ball is rolling too fast
friction force will have opposite sign

Fix by reversing sign of μ

or $$\boxed{t_1 = \frac{2}{7}\frac{v_0}{\mu g}\left|1 - \frac{5h}{2a}\right|}$$

Interesting to get balls final velocity

Plug t_1 in to $\dot{x} = v_0 - \mu g t_1$

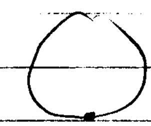
Get $$\boxed{\dot{x}_f = \frac{5}{7} v_0 \left(1 + \frac{h}{a}\right)}$$

if $h = a$, $\dot{x}_f = \frac{10}{7} v_0$, ball speed up

$h = -a$ $\dot{x}_f = 0$, friction brings ball to rest

Shows that placing origin is important

If tried origin = contact point
then friction torque = 0



But $\vec{L}$ changes

$\vec{L}$ of CM = 0 = constant

$\vec{L}$ about CM = spin = changing

Shows $\vec{\Gamma} \neq \frac{d\vec{L}}{dt}$ in general