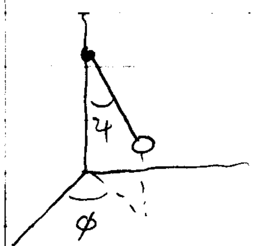
Lecture 14

Again, pick up Foucault problem from last time

Using Lagrangian



$$\vec{\omega} = \omega_e (\cos\theta\,\hat{z} - \sin\theta\,\hat{x})$$

$$\equiv \omega_\perp \hat{z} - \omega_\parallel \hat{x}$$

Got $L = \frac{1}{2} m v^2 + \vec{\omega} \cdot \vec{L} - V$

For $\vec{L} = m \vec{r} \times \vec{v}$

$$L_z = m l^2 \dot{\phi} \sin^2\psi$$

$$L_x = m l^2 [\dot{\psi} \sin\phi + \dot{\phi} \cos\psi \sin\psi \cos\phi]$$

So

$$\boxed{\begin{aligned} L = {} & \tfrac{1}{2} m l^2 (\dot{\psi}^2 + \dot{\phi}^2 \sin^2\psi) + m g l \cos\psi \\ & + m l^2 \omega [\dot{\phi}(\cos\theta \sin^2\psi - \sin\theta \sin\psi \cos\psi \cos\phi) \\ & \quad - \dot{\psi} \sin\theta \sin\phi] \end{aligned}}$$

Look at equation for ϕ:

$$\frac{\partial L}{\partial \dot{\phi}} = m l^2 \dot{\phi} \sin^2\psi + m l^2 \omega (\cos\theta \sin^2\psi - \sin\theta \sin\psi \cos\psi \cos\phi)$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}}\right] = m l^2 \frac{d}{dt}\left[(\dot{\phi} + \omega_\perp) \sin^2\psi\right]$$

$$- m l^2 \omega_\parallel [\dot{\psi} (\cos^2\psi - \sin^2\psi) \cos\phi - \dot{\phi} \sin\psi \cos\psi \sin\phi]$$

and $\frac{\partial L}{\partial \phi} = -m l^2 \omega \sin\theta \left[-\dot{\phi} \sin\psi \cos\psi \sin\phi + \dot{\psi} \cos\phi \right]$

Equation of motion is then

$$\frac{d}{dt}\left[\sin^2\psi\,(\dot{\phi} + \omega_\perp)\right] - \omega_{\parallel}\Big[\dot{\psi}\,(\cos^2\psi - \sin^2\psi)\cos\phi - \dot{\phi}\cos\psi\sin\psi\sin\phi + \dot{\phi}\cos\psi\sin\psi\sin\phi - \dot{\psi}\,(\cos^2\psi + \sin^2\psi)\cos\phi\Big] = 0$$

$$\frac{d}{dt}\left[\sin^2\psi\,(\dot{\phi} + \omega_\perp)\right] - 2\omega_{\parallel}\,\dot{\psi}\sin^2\psi\cos\phi = 0$$

So far, no approximations

But now assume small oscillations: $\psi \ll 1$

$\sin\psi \to \psi$

$$\underbrace{\frac{d}{dt}\left[\psi^2(\dot{\phi} + \omega_\perp)\right]}_{2^{nd} \text{ order in } \psi} - \underbrace{2\omega_{\parallel}\,\dot{\psi}\,\psi^2\cos\phi}_{\substack{3^{rd} \text{ order in } \psi \\ \to \text{ neglect}}} = 0$$

Leaves

$$\psi^2(\dot{\phi} + \omega_\perp) = \text{constant}$$

If $\psi = 0$ at some point in motion,
then constant $= 0$

$$\Rightarrow \boxed{\dot{\phi} = -\omega_\perp = -\omega_e \cos\Theta}$$

Same as before ✓ Note approximation much clearer this way

Test covers material up to here:

Newtonian Mechanics (Ch 1)
- Center of mass & relative coords
- Orbits
- Scattering

Lagrangian Mechanics (Ch 3)
- Hamilton's principle
- Variational calculus
- Eqn of motion
- Small oscillations (simple & complex)
- Continuous media
- Hamiltonian & conservation laws

Non-inertial frame (Ch 2)
- Forces
- Motion on earth
- Lagrangian methods

Probably four questions, one from each chapter

Exam:
- 75 minutes
- Closed book/notes
- I'll supply any non-trivial integrals & trig identities

Historically, my exams average ~50%
- Don't expect to get everything

All problems equal weight, probably not same difficulty
- Do easy ones first!

Start discussing Ch 5 material: Rigid Bodies

Still mostly about rotations

Here rotation isn't specified, need to solve for it

What is a rigid body?

Distribution of mass with <u>fixed</u> shape

Mass density $\rho(\vec{r})$

(or point particles m_p at $\vec{r}_p$)

Parts don't move relative to each other

Body moves as whole

Natural to introduce body coords

= coord frame attached to body

So all body-frame velocities = 0

(no Coriolis force)

Solve for motion of frame

6 coords

3 translation: position

3 angles: orientation

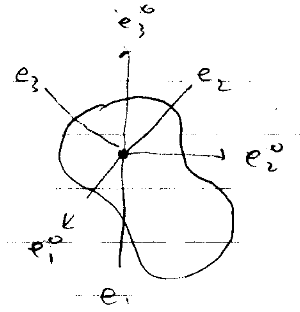
Set up coords + find eqns of motion

Start with rotations:

Assume one point on body is fixed in inertial frame

Take as origin of body $\{\hat{e}_i\}$

+ inertial $\{\hat{e}_i^0\}$



Still have

$$\left(\frac{d\vec r}{dt}\right)_{\text{inert}} = \left(\frac{d\vec r}{dt}\right)_{\text{body}} + \vec\omega\times\vec r$$

But $\left(\frac{d\vec r}{dt}\right)_{\text{body}} = 0$

So $\left(\frac{d\vec r}{dt}\right)_{\text{inert}} = \vec\omega\times\vec r$

and kinetic energy is

$$T = \frac{1}{2}\int\rho(\vec r)\left(\frac{d\vec r}{dt}\right)^2 d^3r$$

$$= \frac{1}{2}\int\rho\,(\vec\omega\times\vec r)\cdot(\vec\omega\times\vec r)\,d^3r$$

use $(A\times B)\cdot(C\times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$

$$T = \frac{1}{2}\int\rho\left[\omega^2 r^2 - (\vec\omega\cdot\vec r)^2\right]d^3r$$

Quadratic in ω

Put in standard form $T = \frac{1}{2}\sum_{ij} I_{ij}\,\omega_i\omega_j$:

$$T = \frac{1}{2}\int\rho\left[r^2\sum_i\omega_i^2 - \sum_i\omega_i r_i\sum_j\omega_j r_j\right]d^3r$$

$$= \frac{1}{2}\sum_{ij}\omega_i\omega_j\int\rho\left[r^2\delta_{ij} - r_i r_j\right]d^3r$$

So $\boxed{I_{ij} = \int\rho\,(r^2\delta_{ij} - r_i r_j)\,d^3r}$

"inertia tensor" 3×3 matrix $\underline{\mathbb{I}}$

(identity = $\underline{\mathbb{1}}$)

Other quantity we'll use a lot is angular momentum $\vec{L}$

$$\vec{L} = \int \rho(\vec{r})\, \vec{r} \times \vec{v}\, d^3r$$

$$= \int \rho(\vec{r})\, \vec{r} \times (\vec{\omega} \times \vec{r})\, d^3r$$

$$A \times (B \times C) = (A \cdot C) B - (A \cdot B) C$$

$$= \int \rho(\vec{r}) \left[r^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}) \vec{r} \right] d^3r$$

$$L_i = \int \rho \left[r^2 \omega_i - \sum_j \omega_j r_i r_j \right] d^3r$$

$$= \sum_j \omega_j \int \rho \left[r^2 \delta_{ij} - r_i r_j \right] d^3r$$

$$= \sum_j I_{ij} \omega_j$$

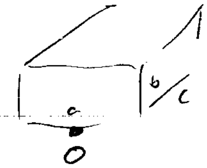
Inertia tensor again

Can see $\underline{\underline{I}}$ is important

Talk a bit about how to evaluate

Basic example: rectangular bar, sides a, b, c

Origin = mid-point of side a



Total mass M

volume $V = abc$

so $\rho = \frac{M}{V} = \frac{M}{abc}$

$$I_{ij} = \rho \int_{-a/2}^{a/2} dx \int_0^b dy \int_0^c dz \left(\delta_{ij} r^2 - r_i r_j \right)$$

$$r^2 = x^2 + y^2 + z^2$$

$$r_1 = x \quad r_2 = y \quad r_3 = z$$

Need integral of $x^2,\ y^2,\ z^2;\ xy,\ xz,\ yz$

$$x^2:\quad \int_{-a/2}^{a/2} x^2\,dx \int_0^b dy \int_0^c dz = \left.\frac{x^3}{3}\right|_{-a/2}^{a/2} bc = \frac{1}{12}a^3bc$$

$$y^2:\quad \int_{-a/2}^{a/2} dx \int_0^b y^2\,dy \int_0^c dz = a\left.\frac{y^3}{3}\right|_0^b c = \frac{1}{3}ab^3c$$

Similar $z^2 \to \frac{1}{3}abc^3$

$$xy:\quad \underbrace{\int_{-a/2}^{a/2} x\,dx}\ \int_0^b y\,dy \int_0^c dz = 0$$

Similar $xz \to 0$

$$yz:\quad \int_{-a/2}^{a/2} dx \int_0^b y\,dy \int_0^c z\,dz = a\,\frac{b^2}{2}\cdot\frac{c^2}{2} = \frac{1}{4}ab^2c^2$$

So $$I_{xx} = \frac{M}{abc}\iiint d^3r\,(x^2+y^2+z^2-x^2)$$

$$= \frac{M}{abc}\left[\frac{1}{3}ab^3c + \frac{1}{3}abc^3\right]$$

$$I_{xx} = \frac{1}{3}M(b^2+c^2)$$

$$I_{yy} = \frac{M}{abc}\iiint d^3r\ x^2+z^2$$

$$= \frac{M}{abc}\left(\frac{1}{12}a^3bc + \frac{1}{3}abc^3\right)$$

$$= \frac{1}{3}M\left(\frac{1}{4}a^2+c^2\right)$$

and $$I_{zz} = \frac{1}{3}M\left(\frac{1}{4}a^2+b^2\right)$$

$$I_{xy} = \frac{M}{abc}\iiint -xy\,d^3r = 0 = I_{xz}$$

$$I_{yz} = \frac{M}{abc} \iiint -yz \, d^3r$$

$$= -\frac{M}{abc}\left(\frac{1}{4} a b^2 c^2\right) = -\frac{1}{4} M bc$$

Clearly $I_{ij} = I_{ji}$

So

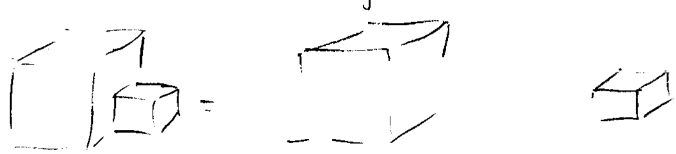
$$\mathbb{I} = M \begin{bmatrix} \frac{b^2+c^2}{3} & 0 & 0 \\ 0 & \frac{a^2}{12} + \frac{c^2}{3} & -\frac{1}{4} bc \\ 0 & -\frac{1}{4} bc & \frac{a^2}{12} + \frac{b^2}{3} \end{bmatrix}$$

For more complicated object, use tricks

First: composition

If object composed of simpler objects j

$$\mathbb{I} = \Sigma \mathbb{I}_j$$



Can even subtract $\mathbb{I}$s to account for holes

* but need to use same origin for all parts!

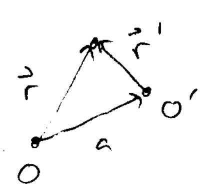
origin = center of rotation

Fortunately, it's easy to relate $\mathbb{I}$ at different origins

Say origin for $\mathbb{I}$ is O

$\mathbb{I}'$ is $O + \vec{a}$

Then $\vec{r}' = \vec{r} - \vec{a}$



$$I'_{ij} = \int d^3r' \, \rho(\vec{r}') \left[\delta_{ij} \, r'^2 - r'_i r'_j \right]$$

$$= \int d^3r \, \rho(\vec{r}) \left[\delta_{ij} |\vec{r} - \vec{a}|^2 - (r_i - a_i)(r_j - a_j) \right]$$

$$= \int d^3r \, \rho(\vec{r}) \left[\delta_{ij} (r^2 - 2\vec{r}\cdot\vec{a} + a^2) - (r_i r_j - r_i a_j - r_j a_i - a_i a_j) \right]$$

$$= \int d^3r \, \rho(\vec{r}) \left[\delta_{ij} r^2 - r_i r_j \right]$$

$$+ (\delta_{ij} a^2 - a_i a_j) \int d^3r \, \rho(\vec{r})$$

$$- 2 \delta_{ij} \vec{a} \cdot \left[\int d^3r \, \rho(\vec{r}) \, \vec{r} \right]$$

$$+ a_i \int d^3r \, \rho(\vec{r}) \, r_j + a_j \int d^3r \, \rho(\vec{r}) \, r_i$$

Note $\int d^3r \, \rho(\vec{r}) \, \vec{r}$ = center of mass $\vec{R}$

So $I'_{ij} = I_{ij} + M(\delta_{ij} a^2 - a_i a_j) - 2\delta_{ij} \vec{a}\cdot\vec{R} + a_i R_j + a_j R_i$

Particularly convenient if origin O is at center of mass

Then $\vec{R} = 0$ and

$$\boxed{I'_{ij} = \bar{I}_{ij} + M(\delta_{ij} a^2 - a_i a_j)}$$ Parallel axis theorem

$\bar{I}_{ij}$ = inertia tensor relative to CoM

Usually best to calculate $\bar{I}_{ij}$, use theorem to translate